Multifractal formalism for self-similar bridges

1998 J. Phys. A: Math. Gen. 31 2567

(http://iopscience.iop.org/0305-4470/31/11/008)

J. Phys. A: Math. Gen. **31** (1998) 2567–2590. Printed in the UK PII: S0305-4470(98)87613-0

Multifractal formalism for self-similar bridges

T Huillet† and B Jeannet

LIMHP-CNRS, Université Paris XIII, Institut Galilée, 93430 Villetaneuse, France

Received 15 September 1997, in final form 13 January 1998

Abstract. We derive the thermodynamics of self-similar paths (or bridges) joining the two points $A := (0, 0)$ and $B := (1, 1)$ of the plane. These paths may be constituted with both macroscopic and microscopic fragments, each deserving its specific statistics, while remaining continuous. Such discontinuous paths are also studied with some information related to the statistics of their jumps.

If the bridges under study are bound to be non-decreasing (A, B)-paths, this study coincides with the one of multifractal measures on the unit interval. Relaxing this condition leads to an extension of the multifractal formalism whose main lines are derived here.

1. Introduction

Multifractal measures on the unit interval and their associated spectra have been designed by physicists [1–3] in order to understand problems arising in the natural sciences: in fully developed turbulence, they allow the investigation of the intermittent behaviour in the high-vorticity domain; in diffusion limited aggregation, they are designed to compute the probability that a random walker hits the neighbourhood of a particular site of an aggregate [4, 5]. They are also studied in dynamical system theory to measure how often a given region of the attractor is visited.

In terms of the equivalent distribution function of such measures, their graph is bound to be a non-decreasing (A, B)-path, joining the two points $A := (0, 0)$ and $B := (1, 1)$ of the plane.

We first derive, in sections 2–4, the thermodynamics of such self-similar paths (or bridges) joining the two points A and B of the plane. These paths may be constituted of both microscopic (singular) and macroscopic (regular) fragments, each deserving its specific statistics, while first remaining continuous: in the first 'diluted' regime, the partition function of the fragments is renormalizable, while it converges in the second 'condensed' regime; a phase transition separates these regimes. In the condensed regime, we propose a description of the regular part of these objects, based on Gibbs' statistics. Such discontinuous paths are also studied with information related to the statistics of their jumps.

Relaxing the condition that the path should be non-decreasing leads to an extension of the multifractal formalism for measures, the main lines of which are derived in section 5. In this extension, (A, B)-paths are the outcome of a deterministic conservative multiplicative cascade whose reduction factors are allowed to be negative. As a result, the partition function encloses information on both their moduli and sign. Circumventing these difficulties, the (A, B)-paths that are designed here are truly multifractals.

† E-mail address: huillet@limhp.univ-paris13.fr

0305-4470/98/112567+24$19.50

2. Multifractal formalism

Let us first recall some well known facts from multifractal theory. Let μ be a Borel probability measure supported by the interval $[0,1]$. The measure μ is said to be discontinuous (or atomic) if it contains atoms, in which case there exists at least one singleton $x_0 \in [0,1]$ for which $\mu(\{x_0\}) > 0$. If it is not discontinuous, it is said to be continuous (or diffuse). If it is continuous, it can be absolutely continuous if the density $\lim_{\varepsilon\to 0^+}(1/\varepsilon)\mu(B_\varepsilon(x)) := p(x) \geqslant 0$ exists for $x \in (0,1)$ and $B_\varepsilon(x) := [x-\varepsilon, x+\varepsilon]$; otherwise it is singular continuous. Moreover, it may have gaps, in which case there is at least one open sub-interval $I \subset [0,1]$, for which $\mu(I) = 0$. Mixtures of all these situations may arise, since each such measure can be (uniquely) decomposed into a purely atomic, a continuous but singular, and an absolutely continuous part.

In any case, it is first assumed that the distribution function $x \in [0,1] \to F(x) := \mu([0,x]) \in [0,1]$ is a right-continuous, non-decreasing function, and that $\mu([0,1]) = 1$.

For any $x \in [0,1]$, now let

$$\alpha(x) := \lim_{\epsilon\to 0+} \log_\varepsilon \mu(B_\varepsilon(x)) \tag{2.1}$$

be the local Hölder exponent of μ at x. If μ has gaps, we shall adopt the convention $\alpha(x_1) = +\infty$, for any x_1 within an open interval I, such that $\mu(I) = 0$. In contrast, $\alpha(x_0) = 0$ if μ is atomic at x_0. Let $K(\alpha) := \{x : \alpha(x) = \alpha\}$, for $\alpha \geqslant 0$, and $f_{\mathrm{H}}(\alpha) \in [0,1]$ be the Hausdorff dimension of the set $K(\alpha)$. The function $f_{\mathrm{H}}(\alpha)$ is called the Hausdorff spectrum of μ. It is said that μ is a multifractal measure if the support of the function $f := \{\alpha \geqslant 0 : f(\alpha) \neq 0\}$ contains a continuum of α.

Thus a multifractal is a measure. It is finely characterized by the function f_{H} which gives the Hausdorff dimension of the fractal sets $K(\alpha)$, as $\alpha \geqslant 0$ varies. This function is, in general, inaccessible in most cases, except for self-similar measures which we shall limit ourselves to in the following. In this case, the Legendre spectrum also plays a central role.

3. The purely singular case

We first recall the construction of a purely singular measure, without gaps [6].

A continuous self-similar measure can classically be constructed in the following way, taking advantage of the notion of a multiplicative cascade. Take initially a unit mass uniformly spread over the interval $[0,1]$. At resolution $n = 1$, split the unit interval into m sub-intervals each in the similarity ratio $r_l, l = 1, \ldots, m$, with $r_l \in (0,1)$, $l = 1, \ldots, m$, such that $\sum_{l=1}^m r_l = 1$. Attribute the mass $\pi_l, l = 1, \ldots, m$, uniformly to all such sub-intervals. Here $\pi_l \in (0,1)$, $l = 1, \ldots, m$, satisfying $\sum_{l=1}^m \pi_l = 1$ which expresses the mass conservation. At resolution $n = 2$, split each sub-interval $l = 1, \ldots, m$ into sub-sub-intervals, each in the similarity ratio $r_{l'}, l' = 1, \ldots, m$, to the chosen sub-interval of length r_l, and attribute the mass $\pi_l\pi_{l'}, l' = 1, \ldots, m$, uniformly to each such sub-sub-interval. Iterate indefinitely.

In terms of the distribution function, this algorithm starts with the straight line $F_0(x) = x$, which connects the points $(0,0)$ and $(1,1)$. At resolution $n = 1$, $F_1(x)$ is an increasing broken line with m pieces. During iteration, the graph of F_n becomes more and more tortuous, while remaining a continuous and strictly increasing path (or bridge) from $A := (0,0)$ to $B := (1,1)$.

The limit result $F := \lim_{n\to\infty} F_n$ is the distribution function of a continuous purely singular measure μ spread over the unit interval.

In this sense, the following formalism is the thermodynamics of these particular (A, B)-paths (or bridges).

For $(q, q^*) \in \mathbb{R}^2$, define the level-$n$ partition function

$$Z_n(q, q^*) := \sum_{i=1}^{N_n} \pi_i(n)^q r_i(n)^{q^*} \qquad Z_n(0, 0) = N_n \tag{3.1}$$

where $(\pi_i(n), r_i(n))_{i=1,\dots,N_n}$ is the vector of masses and similarity ratios attached to each of the $N_n := m^n$ available chunks at resolution n.

Clearly, $Z_n(q, q^*)$ is defined recursively by

$$Z_{n+1}(q, q^*) = Z_n(q, q^*) z(q, q^*) \qquad Z_0(q, q^*) = 1 \tag{3.2}$$

with

$$z(q, q^*) := \sum_{l=1}^{m} \pi_l^q r_l^{q^*}.$$

Next define $F(q, q^*) := -\log_m z(q, q^*)$. This function is concave and analytic in the plane, satisfying $F(0, 0) = -1$ and $F(0, 1) = F(1, 0) = 0$.

It follows from (3.1) and (3.2) that for any $n \geqslant 1$

$$-\log_{N_n} Z_n(q, q^*) = F(q, q^*). \tag{3.3}$$

Define now, as in [1], the function $\tau(q)$ implicitly by

$$\sum_{l=1}^{m} \pi_l^q r_l^{-\tau(q)} = 1. \tag{3.4}$$

Alternatively, $F(q, -\tau(q)) = 0$. Observe that $Z_n(q, q^*) \to \infty$ if $q^* < -\tau(q)$ and tends to zero otherwise.

The function $\tau(q)$ is concave and analytic on the line, satisfying $\tau(0) = -1$ and $\tau(1) = 0$. Its Legendre transform

$$f(\alpha) := \inf_{q \in \mathbb{R}} (\alpha q - \tau(q)) = \alpha f'(\alpha) - \tau(f'(\alpha)) \tag{3.5}$$

is defined on the Hölder range $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, with $\alpha_{\min} := \min_{l=1,\dots,m}(-\log \pi_l / \log r_l) > 0$ and $\alpha_{\max} := \max_{l=1,\dots,m}(-\log \pi_l / -\log r_l) > 0$. This function is called the Legendre spectrum of the measure μ: it is non-negative, concave and analytic on this interval, with the properties $f(\alpha_{\min}) = f(\alpha_{\max}) = 0$ and $f'(\alpha_{\min}) = f'(\alpha_{\max}) = +\infty$. It attains its maximum at $\alpha_0 := (1/m) \sum_{l=1}^{m} (\log \pi_l / \log r_l) = \tau'(0)$, and $f(\alpha_0) = 1$. The point $\alpha_1 := \tau'(1)$, for which $\alpha_1 = f(\alpha_1)$ is called the information dimension of μ.

In this context, the function $f(\alpha)$ is of some importance, since it coincides with the Hausdorff spectrum of the self-similar measure μ just mentioned. In other words, $f(\alpha) = f_{\mathrm{H}}(\alpha)$.

More precisely, define the set

$$B(\alpha, \gamma^*) := \left\{ i \in [N_n] : \frac{-\log_{N_n} \pi_i(n)}{-\log_{N_n} r_i(n)} \to \alpha, -\log_{N_n} r_i(n) \to \gamma^* \right\} \tag{3.6}$$

and $N_n(\alpha, \gamma^*) := \#B(\alpha, \gamma^*)$ its cardinal. This is the number of atoms whose coarse Hölder exponent is α *and* whose similarity ratio grows like $N_n^{-\gamma^*}(\gamma^* > 0)$. We first learn from large deviation theory [7, 8] that

$$\lim_{n \to \infty} \log_{N_n} N_n(\alpha, \gamma^*) = f_\alpha(\gamma^*) \tag{3.7}$$

where $f_\alpha(\gamma^*) := f(\alpha\gamma^*, \gamma^*)$ and

$$f(\gamma, \gamma^*) := \inf_{(q,q^*)\in\mathbb{R}^2} (\gamma q + \gamma^* q^* - F(q, q^*)).$$

The function $f_\alpha(\gamma^*)$ possesses the following properties [8]. For any $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, it is positive concave as a function of γ^*. It takes the maximal value $\sigma(\alpha) := f_\alpha(0)$. The line passing through the origin which is tangent to the γ^*-graph of $f_\alpha(\gamma^*)$ has slope $f(\alpha)$ and hits this graph at $\gamma^* := \gamma^*(\alpha)$. Hence, $f(\alpha) = f_\alpha(\gamma^*(\alpha))/\gamma^*(\alpha)$.

Now, for any $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ held fixed, the contribution to the partition function $Z_n(q, q^*)$ by those chunks with this Hölder exponent *and* whose similarity ratio is $N_n^{-\gamma^*}$ at resolution n is from (3.6) and (3.7)

$$Z_n(q, q^*; \alpha, \gamma^*) \underset{n\to\infty}{\sim} N_n^{f_\alpha(\gamma^*)} N_n^{-\alpha\gamma^* q} N_n^{-\gamma^* q^*} = N_n^{f_\alpha(\gamma^*) - \gamma^*(\alpha q + q^*)}$$

(the product of their number times their (mass)^q(volume)$^{q^*}$). This contribution will be maximal for the unique optimal similarity ratio $\gamma^*_{\rm op}$ satisfying $f'_\alpha(\gamma^*_{\rm op}) = (\alpha q + q^*)$ and will be $Z_n(q, q^*; \alpha, \gamma^*_{\rm op}) \sim N_n^{f_\alpha(\gamma^*_{\rm op}) - \gamma^*_{\rm op} f'_\alpha(\gamma^*_{\rm op})}$.

If $q^* > -\tau(q)$, $f_\alpha(\gamma^*_{\rm op}) - \gamma^*_{\rm op} f'_\alpha(\gamma^*_{\rm op}) < 0$: the maximal contribution of $B(\alpha, \gamma^*)$ to $Z_n(q, q^*)$ tends to zero, together with $Z_n(q, q^*)$ itself.

If $q^* < -\tau(q)$, $f_\alpha(\gamma^*_{\rm op}) - \gamma^*_{\rm op} f'_\alpha(\gamma^*_{\rm op}) > 0$: the maximal contribution of $B(\alpha, \gamma^*)$ to $Z_n(q, q^*)$ tends to infinity, together with $Z_n(q, q^*)$ itself.

If $q^* = -\tau(q)$, $f_\alpha(\gamma^*_{\rm op}) - \gamma^*_{\rm op} f'_\alpha(\gamma^*_{\rm op}) < 0$: the maximal contribution of $B(\alpha, \gamma^*)$ to $Z_n(q, q^*)$ also tends to zero, unless $q = f'(\alpha)$. Here, $\gamma^*_{\rm op} = \gamma^*(\alpha)$ and

$$f_\alpha(\gamma^*_{\rm op}) - \gamma^*_{\rm op} f'_\alpha(\gamma^*_{\rm op}) = f_\alpha(\gamma^*(\alpha)) - \gamma^*(\alpha)(\alpha f'(\alpha) - \tau(f'(\alpha)))$$

which is, from the above analysis, $f(\alpha)\gamma^*(\alpha) - \gamma^*(\alpha) f(\alpha) = 0$. In this case only, the maximal contribution of $B(\alpha, \gamma^*)$ to $Z_n(q, q^*)$ is one: the set $B(\alpha, \gamma^*(\alpha))$ carries all the information $Z_n(q, -\tau(q)) = 1$. The atoms of the α-similarity ratio $r_n(\alpha) := N_n^{-\gamma^*(\alpha)}$ therefore carry all the information and we get from (3.7)

$$\lim_{n\to\infty} \log_{1/r_n(\alpha)} N_n(\alpha, \gamma^*(\alpha)) = f(\alpha) \tag{3.8}$$

which is consistent with the definition of a Hausdorff dimension in our situation.

Remark 1. Observe now that another function $\tau^*(q^*)$ could be defined by $F(-\tau^*(q^*), q^*) = 0$, with the same properties as τ [9–11]. Clearly $-\tau^*(-\tau(q)) = q$ and $-\tau(-\tau^*(q^*)) = q^*$. These functions are the inverses of one another. As a result, letting $f^*(\alpha^*) := \inf_{q^*\in\mathbb{R}}(\alpha^* q^* - \tau^*(q^*))$ be the Legendre transform of τ^*, it is well known that $f^*(\alpha^*) = \alpha^* f(1/\alpha^*)$ and conversely that $f(\alpha) = \alpha f^*(1/\alpha)$. The function $f^*(\alpha^*)$ is the Legendre spectrum of the measure μ^* obtained while reversing the roles played by $(\pi_l, r_l)_{l=1,\ldots,m}$ in the above construction of μ. The measures μ and μ^* are said to be reciprocal.

Remark 2. Various extensions and 'anomalies' of this basic model have been proposed recently in the literature about multifractals:

- the infinite base case: $m = \infty$, leading to the notion of a left-hand sided Legendre spectrum [4, 12, 13];
- a randomization procedure of $(\pi_l, r_l)_{l=1,\ldots,m} \to (\Pi_l, R_l)_{l=1,\ldots,m}$, as identically distributed but not independent variables on the interval $[0, 1]$ (as a result of exact mass and volume conservation $\sum_l \Pi_l = 1$ and $\sum_l R_l = 1$), leading to the notion of random multifractals, for which negative ('latent') $f(\alpha)$ are observed and interpreted [14–16];

• a randomization procedure of $(\pi_l, r_l)_{l=1,\ldots,m}$, as identically distributed and independent on the *positive* real line (in particular, log-normal) but satisfying the weaker average constraint: $\boldsymbol{E}(\sum_l \Pi_l) = \boldsymbol{E}(\sum_l R_l) = 1$, leading to the notion of random multifractals, for which 'virtual' negative α are observed [17];

• the 'skewed' multifractals extension when each fragment splits at various resolutions [18, 19].

4. Mixture of an absolutely continuous measure with a singular continuous measure

First we now construct a continuous self-similar measure which presents *simultaneously* singular and absolutely continuous parts. Brown *et al* [20] have introduced a formalism similar to the one which is discussed in this section. In order to avoid confusion with the purely singular self-similar measures of the previous section, we shall call these measures self-similar 'with condensation'.

Take initially a unit mass uniformly spread over the interval $[0, 1]$. At resolution $n = 1$, split the unit interval into m sub-intervals as before.

Suppose there are now *two* types of sub-intervals. First, the 'sterile' ones $l = 1, \ldots, d < m - 1$ each in the similarity ratio $r_{l,-}, l = 1, \ldots, d$, with $r_{l,-} \in (0, 1)$, $l = 1, \ldots, d$ such that $\sum_{l=1}^{d} r_{l,-} := r_- < 1$. Attribute the mass $\pi_{l,-} \in (0, 1), l = 1, \ldots, d$, with $\sum_{l=1}^{d} \pi_{l,-} := \pi_- < 1$, uniformly to each such sterile sub-interval. These sub-intervals are sterile in the sense that they will no longer split in any subsequent step.

The second type of sub-intervals are 'productive', each in the similarity ratio $r_{l,+} \in (0, 1), l = d + 1, \ldots, m$, with $\sum_{l=d+1}^{m} r_{l,+} := r_+ = 1 - r_-$. There are thus $b := m - d > 1$ such sub-intervals. Attribute the mass $\pi_{l,+} \in (0, 1), l = d + 1, \ldots, m$, uniformly to each such productive sub-interval. Suppose $\sum_{l=d+1}^{m} \pi_{l,+} := \pi_+ = 1 - \pi_-$. It is then assumed that the mass and volume conservations hold, globally. At resolution $n = 2$, split each productive sub-interval $l = d + 1, \ldots, m$ leaving the sterile ones unaffected. Upon indefinite iteration, we are also left with a singular measure μ spread over the unit interval. However, in the limit, macroscopic masses remain supported by macroscopic volumes and co-exist with the singular part of the distribution, each corresponding to a specific thermodynamical regime (condensed and diluted, respectively). We now derive these two formalisms adapted to each situation.

4.1. Singular part of μ

For $(q, q^*) \in \mathbb{R}^2$, define as before the level-n partition function

$$Z_n(q, q^*) := \sum_{i=1}^{N_n} \pi_i(n)^q r_i(n)^{q^*}$$

where $(\pi_i(n), r_i(n))_{i=1,\ldots,N_n}$ is the vector of masses and similarity ratios attached to each of the N_n available chunks at resolution n.

Clearly, $Z_n(q, q^*)$ is now defined recursively by

$$Z_{n+1}(q, q^*) = z_-(q, q^*) + Z_n(q, q^*) z_+(q, q^*) \qquad Z_0(q, q^*) = 1 \tag{4.1}$$

with

$$z_-(q, q^*) := \sum_{l=1}^{d} \pi_{l,-}^q r_{l,-}^{q^*} \qquad \text{and} \qquad z_+(q, q^*) := \sum_{l=d+1}^{m} \pi_{l,+}^q r_{l,+}^{q^*}.$$

Observe from (4.1) that $Z_n(0,0) := N_n$ satisfies

$$N_{n+1} = d + N_n b \qquad N_0 = 1.$$

Next define formally the generating function $Z(s,q,q^*) := \sum_{n\geqslant 0} b^{sn} Z_n(q,q^*)$. It follows from (4.1) that

$$Z(s,q,q^*) = \frac{1-b^s(1-z_-(q,q^*))}{(1-b^s)(1-b^s z_+(q,q^*))} \tag{4.2}$$

provided $s \leqslant F(q,q^*) := \min(0, F_+(q,q^*))$, with $F_+(q,q^*) := -\log_b z_+(q,q^*)$, and is undefined otherwise.

We shall let $D := \{(q,q^*) \in \mathbb{R}^2 : F_+(q,q^*) < 0\}$, and D^{c} be its complement in $\mathbb{R}^2$.

Now, the function F_+ is concave and analytic in the plane, satisfying $F_+(0,0) = -1$, $F_+(0,1) = -\log_b r_+ > 0$ and $F_+(1,0) = -\log_b \pi_+ > 0$.

The function F also is concave but only continuous, and this is the signature of a *phase transition* between the diluted (when the scaling exponents (q,q^*) belong to D) and condensed phases (when $(q,q^*) \in D^{\mathrm{c}}$) [11, 18, 21].

It follows from (4.2) that

$$\lim_{n\to\infty} -\log_{N_n} Z_n(q,q^*) = F(q,q^*) \tag{4.3}$$

generalizing (3.3).

Now define now the function $\tau(q)$ implicitly by

$$F_+(q, -\tau(q)) = 0. \tag{4.4}$$

To make the analogy of this definition of the 'structure function' $\tau(q)$ with those available in related papers more transparent, it is equivalently defined by

$$\sum_{l=d+1}^{m} \pi_{l,+}^{q} r_{l,+}^{-\tau(q)} = 1.$$

This is the equation of the critical line separating D and D^{c}.

This function $\tau : \mathbb{R} \to \mathbb{R}$ is concave and analytic, with $0 > \tau(0) > -1$ and $\tau(q_+) = 0$, where $q_+ \in (0,1)$ is the unique real number defined by $\sum_{l=d+1}^{m} \pi_{l,+}^{q_+} = 1$. Observe that $\tau(0)$ is uniquely determined by $\sum_{l=d+1}^{m} r_{l,+}^{-\tau(0)} = 1$ in a similar way.

The Legendre transform $f(\alpha) := \inf_{q\in\mathbb{R}}(\alpha q - \tau(q))$ is defined on the Hölder range $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, with $\alpha_{\min} := \min_{l=d+1,\ldots,m}(-\log \pi_{l,+}/ -\log r_{l,+}) > 0$ and $\alpha_{\max} := \max_{l=d+1,\ldots,m}(-\log \pi_{l,+}/ -\log r_{l,+}) > 0$. This function is called the Legendre spectrum of the singular part of measure μ. The function $f(\alpha)$ is non-negative, concave and analytic on this interval, with $f(\alpha_{\min}) = f(\alpha_{\max}) = 0$ and $f'(\alpha_{\min}) = f'(\alpha_{\max}) = +\infty$. It attains its maximum at $\alpha_0 := (1/b)\sum_{l=d+1}^{m}(\log \pi_{l,+}/\log r_{l,+})$, and $f(\alpha_0) = -\tau(0) < 1$, showing that macroscopic masses remain supported by macroscopic volumes.

At the point $\alpha_1 := \tau'(q_+) < \alpha_0$, $f(\alpha_1) = q_+\tau'(q_+)$. This number is called the information dimension of the singular part of μ.

In this context, the function $f(\alpha)$ coincides with the Hausdorff spectrum of the singular part of the self-similar measure μ just constructed, which means that $f(\alpha) = f_{\mathrm{H}}(\alpha)$. (The proof of this assertion can, for example, be found while readily adapting [10, theorem 16, p 13], which itself is based on [22] concerning random multifractals; the main source of the difficulty here is that $\sum_{l=d+1}^{m} \pi_{l,+} < 1$, which may be circumvented, with additional complexities to overcome, while performing the substitution $\pi_{l,+} \to \pi_{l,+}(q) := \pi_{l,+}^{q} r_{l,+}^{-\tau(q)}$ for which now $\sum_{l=d+1}^{m} \pi_{l,+}(q) = 1$.)

An interpretation of this function, which follows step by step that given at the end of section 3, can easily be derived.

We now come to the thermodynamics, of the regular part of μ.

4.2. Regular part of μ

First note that really equation (4.3) is only meaningful for $(q, q^*) \in D$.

If $(q, q^*) \in D^c$, $\lim_{n\to\infty} -\log_{N_n} Z_n(q, q^*) = 0$ still holds but we have a more interesting limit result,

$$\lim_{n\to\infty} -\log Z_n(q, q^*) = -\log(\lim_{s\to 0-} (1 - b^s)Z(s, q, q^*)) := F_\infty(q, q^*)$$

with $F_\infty(q, q^*) := -\log(z_-(q, q^*)/(1 - z_+(q, q^*)))$, from the initial-value theorem of Laplace. Thus, concerning the regular (that is absolutely continuous) part of the measure μ, we observe that the partition function converges in D^c, which is the condensed phase region.

Next define $\tau_\infty(q)$ by

$$F_\infty(q, -\tau_\infty(q)) = 0$$

hence

$$\sum_{l=1}^{d} \pi_{l,-}^{q} r_{l,-}^{-\tau_\infty(q)} + \sum_{l=d+1}^{m} \pi_{l,+}^{q} r_{l,+}^{-\tau_\infty(q)} = 1 \tag{4.5}$$

is its defining equation (observe the analogy with (3.4)). The graph of $-\tau_\infty$ is entirely included within D^c; indeed, $-\tau_\infty(0) = 1$ and $-\tau_\infty(1) = 0$ so that two points of its graph are at least within D^c. If now $-\tau_\infty$ and $-\tau$ were to intersect, there should exist a point, say q_0, for which $-\tau_\infty(q_0) = -\tau(q_0)$. However, from the equation $\sum_{l=d+1}^{m} \pi_{l,+}^{q} r_{l,+}^{-\tau(q)} = 1$ defining $\tau(q)$ and the one defining $\tau_\infty(q)$, such a q_0 should satisfy

$$\sum_{l=d+1}^{m} \pi_{l,+}^{q_0}(r_{l,+}^{-\tau(q_0)} - r_{l,+}^{-\tau_\infty(q_0)}) = 0 = \sum_{l=1}^{d} \pi_{l,-}^{q_0} r_{l,-}^{-\tau_\infty(q_0)}$$

which is impossible, except maybe for $\tau_\infty(q_0) = +\infty$ and hence at $q_0 = +\infty$. We shall then call

$$f_\infty(\alpha) := \inf_{q\in\mathbb{R}} (\alpha q - \tau_\infty(q)) \tag{4.6}$$

the Hölder spectrum of the regular part of μ.

These facts deserve some explanation.

4.2.1. A Gibbs analysis of the regular part of μ. Write the partition function Z_∞ as $Z_\infty(q, q^*) := \sum_{i\geqslant 1} \mathrm{e}^{-q x_1(i) - q^* x_2(i)}$, with $x_1(i) := -\log \mu(i)$ and $x_2(i) := -\log r(i)$ the logarithms of mass and similarity ratios of the observable macroscopic atoms constituting the regular part of μ (microscopic chunks contribute to nothing in this sum while the scaling exponents (q, q^*) vary in D^c).

Assume now that the exact distribution of $(x_1(i), x_2(i))_{i\geqslant 1}$ is unknown to some observer so that these values are assumed to be the realization of some random vector (X_1, X_2). We shall then search for a 'probability' measure of the event $(X_1 = x_1(i), X_2 = x_2(i))$, say $\boldsymbol{P}(X_1 = x_1(i), X_2 = x_2(i)) := \boldsymbol{P}(x_1(i), x_2(i))$, which maximizes the Shannon entropy [23, 24]:

$$S(\boldsymbol{P}) := -\sum_{i\geqslant 1} \boldsymbol{P}(x_1(i), x_2(i)) \log \boldsymbol{P}(x_1(i), x_2(i)) \tag{4.7}$$

under the constraints

$$\sum_{i\geqslant 1} \boldsymbol{P}(x_1(i), x_2(i)) = 1 \tag{4.8}$$
$$\sum_{i\geqslant 1} x_1(i)\boldsymbol{P}(x_1(i), x_2(i)) = \gamma := \langle X_1\rangle$$

and

$$\sum_{i\geqslant 1} x_2(i)\boldsymbol{P}(x_1(i), x_2(i)) = \gamma^* := \langle X_2\rangle$$

fixing the theoretical averages (γ, γ^*) of (X_1, X_2) under the probability distribution $\boldsymbol{P}$.

Performing this standard optimization program using Lagrange multipliers yields the Gibbs distribution

$$\boldsymbol{P}_{(q,q^*)}(x_1(i), x_2(i)) = \frac{\mathrm{e}^{-qx_1(i)-q^*x_2(i)}}{Z_\infty(q, q^*)} \qquad i \geqslant 1, (q, q^*) \in D^{\mathrm{c}} \tag{4.9}$$

with (q, q^*) and $(\gamma = \langle X_1\rangle, \gamma^* = \langle X_2\rangle)$ related by

$$\partial_q F_\infty(q, q^*) = \gamma \qquad \text{and} \qquad \partial_{q^*} F_\infty(q, q^*) = \gamma^*. \tag{4.10}$$

Thus, a natural model for the probability to observe $(x_1(i), x_2(i))$ is the 'exponential' Gibbs family (4.9), as a function of the 'external' control parameters $(q, q^*) \in D^{\mathrm{c}}$, related to the theoretical averages (γ, γ^*) of the distribution as just mentioned.

Thus, the more visible the object is, through the *joint* information $mass^q volume^{q^*}$, the larger the probability of this observation will be.

This actually is one of the postulates of statistical physics. Statistics is then concerned with the problem of identifying the value of (q, q^*) which fits the observation sample the best.

4.2.2. The maximum likelihood estimator of the scaling exponents. We shall recall how to construct a maximum likelihood estimator of (q, q^*).

The log-Laplace transform $F_\infty(q, q^*)$ is concave on the convex set D^{c}. Its Legendre transform

$$S(\gamma, \gamma^*) := \inf_{(q,q^*)\in D^{\mathrm{c}}} (\gamma q + \gamma^* q^* - F_\infty(q, q^*)) \tag{4.11}$$

is well defined, non-negative and concave on the convex hull of $(x_1(i), x_2(i))_{i\geqslant 1}$.

Moreover, $S(\gamma, \gamma^*) = \gamma\partial_\gamma S + \gamma^*\partial_{\gamma^*} S - F_\infty(\partial_\gamma S, \partial_{\gamma^*} S)$, with

$$(\langle X_1\rangle, \langle X_2\rangle) = (\gamma, \gamma^*).$$

Conversely, the control parameters (q, q^*) can be derived from (γ, γ^*) by $q = \partial_\gamma S$ and $q^* = \partial_{\gamma^*} S$. The distribution $\boldsymbol{P}_{(q,q^*)}$ is thus well parametrized by (γ, γ^*), through

$$\boldsymbol{P}_{(\gamma,\gamma^*)}(x_1(i), x_2(i)) = \frac{\mathrm{e}^{-\partial_\gamma S.x_1(i)-\partial_{\gamma^*} S.x_2(i)}}{Z_\infty(\partial_\gamma S, \partial_{\gamma^*} S)} \qquad i \geqslant 1. \tag{4.12}$$

Note also that $S(\gamma, \gamma^*) = S(\boldsymbol{P}_{(\gamma,\gamma^*)})$ is the Shannon entropy [25] evaluated at $\boldsymbol{P} = \boldsymbol{P}_{(\gamma,\gamma^*)}$.

A maximum likelihood estimator of (q, q^*), say (Q, Q^*), can therefore be derived from an estimator (Γ, Γ^*) of the theoretical averages (γ, γ^*) by $(Q, Q^*) = (\partial_\gamma S(\Gamma, \Gamma^*), \partial_{\gamma^*} S(\Gamma, \Gamma^*))$.

Now let $(X_1(k), X_2(k))_{k=1}^K$be an independent K-sample of the random variables (X_1, X_2). Introducing the likelihood (i.e. the probability of the K-sample), $V_{(\gamma,\gamma^*)} := \prod_{k=1}^K \boldsymbol{P}_{(\gamma,\gamma^*)}(X_1(k), X_2(k))$, and searching for the value of (γ, γ^*) maximizing this likelihood, we get

$$\overline{X}_1(K) = \frac{1}{K}\sum_{k=1}^K X_1(k) \qquad \text{and} \qquad \overline{X}_2(K) = \frac{1}{K}\sum_{k=1}^K X_2(k).$$

The empirical averages are thus unbiased, efficient estimators of (γ, γ^*), in the sense that the expectation and variance–covariance under $\boldsymbol{P}_{(\gamma,\gamma^*)}$ are

$$(\langle\overline{X}_1(K)\rangle, \langle\overline{X}_2(K)\rangle) = (\gamma, \gamma^*) \qquad \text{and} \qquad \boldsymbol{\Sigma}[\overline{X}_1(K), \overline{X}_2(K)] = -\frac{1}{K}H^{-1}(\gamma, \gamma^*).$$

The quantity $-KH(\gamma, \gamma^*)$ is the Fisher information matrix of the K-sample, with H the Hessian matrix of $S(\gamma, \gamma^*)$.

We also have the law of large numbers

$$(\overline{X}_1(K), \overline{X}_2(K)) \underset{K\to\infty}{\to} (\gamma, \gamma^*) \tag{4.13}$$

with $\boldsymbol{P}_{(\gamma,\gamma^*)}$-probability one, and the central limit theorem

$$\lim_{K\to\infty} \boldsymbol{P}_{(\gamma,\gamma^*)}(\boldsymbol{\Sigma}^{-1/2}((\overline{X}_1(K) - \gamma, \overline{X}_2(K) - \gamma^*)') \leqslant \boldsymbol{y}) = \mathrm{erf}(\boldsymbol{y}) \tag{4.14}$$

together with its large deviation counterpart [26]

$$\frac{1}{K}\log \boldsymbol{P}_{(\gamma,\gamma^*)}(\overline{X}_1(K) \to \gamma_o, \overline{X}_2(K) \to \gamma_o^*) \underset{K\to\infty}{\to} -K(\boldsymbol{P}_{(\gamma_o,\gamma_o^*)} \parallel \boldsymbol{P}_{(\gamma,\gamma^*)}). \tag{4.15}$$

Here (γ_o, γ_o^*) are the observed empirical averages and

$$K(\boldsymbol{P}_{(\gamma_o,\gamma_o^*)} \parallel \boldsymbol{P}_{(\gamma,\gamma^*)}) = \sum_{i\geqslant 1} \boldsymbol{P}_{(\gamma_o,\gamma_o^*)}(x_1(i), x_2(i)) \log \frac{\boldsymbol{P}_{(\gamma_o,\gamma_o^*)}(x_1(i), x_2(i))}{\boldsymbol{P}_{(\gamma,\gamma^*)}(x_1(i), x_2(i))}$$

is the Kullback non-negative information gain between $\boldsymbol{P}_{(\gamma_o,\gamma_o^*)}$ and $\boldsymbol{P}_{(\gamma,\gamma^*)}$. As a result,

$$K(\boldsymbol{P}_{(\gamma_o,\gamma_o^*)} \parallel \boldsymbol{P}_{(\gamma,\gamma^*)}) = S(\gamma, \gamma^*) - S(\gamma_o, \gamma_o^*) - (\gamma - \gamma_o, \gamma^* - \gamma_o^*)\nabla S(\gamma, \gamma^*). \tag{4.16}$$

4.2.3. A large deviation result. Now we come to the main result of this section. We try to evaluate the (small) probability that the empirical averages of a K-sample converge like $\overline{X}_1(K)/\overline{X}_2(K) \to \alpha$ and $\overline{X}_2(K) \to \gamma_o^*$, under the Gibbs probability $\boldsymbol{P}_{\alpha,\gamma^*}$, fixing the theoretical averages $\langle X_1\rangle/\langle X_2\rangle = \alpha$ and $\langle X_2\rangle = \gamma^*$. It follows from (4.15) that

$$\frac{1}{K}\log \boldsymbol{P}_{\alpha,\gamma^*}\left(\frac{\overline{X}_1(K)}{\overline{X}_2(K)} \to \alpha, \overline{X}_2(K) \to \gamma_o^*\right) \underset{K\to\infty}{\to} -K(\boldsymbol{P}_{(\alpha\gamma_o^*,\gamma_o^*)} \parallel \boldsymbol{P}_{(\alpha\gamma^*,\gamma^*)}) \leqslant 0. \tag{4.17}$$

So doing, the ratio of theoretical averages $\langle X_1\rangle/\langle X_2\rangle = \alpha$ is held fixed. Here $\overline{X}_1(K)/\overline{X}_2(K)$ denotes the ratio of the empirical averages of the logarithm of mass and the logarithm of volume.

Now the function $S(\alpha\gamma^*, \gamma^*)$ is non-negative and non-decreasing. The line passing through the origin which is tangent to the γ^*-graph of $S(\alpha\gamma^*, \gamma^*)$ has slope $\partial_{\gamma^*}S(\alpha\gamma^*, \gamma^*) = f_\infty(\alpha)$ defined by (4.6) and hits this graph at $\gamma^* := \gamma^*(\alpha)$. Hence, $f_\infty(\alpha) = S(\alpha\gamma^*, \gamma^*)/\gamma^*(\alpha)$.

It follows from (4.16) and (4.17) that, at $\gamma^* = \gamma^*(\alpha)$,

$$\frac{1}{K}\log \boldsymbol{P}_{\alpha,\gamma^*(\alpha)}\left(\frac{\overline{X}_1(K)}{\overline{X}_2(K)} \to \alpha,\ \overline{X}_2(K) \to \gamma_o^*\right) \underset{K\to\infty}{\to} S(\alpha\gamma_o^*, \gamma_o^*) - \gamma_o^* f_\infty(\alpha) \leqslant 0. \tag{4.18}$$

This rate function has no *explicit* dependence on the theoretical average $\gamma^* = \gamma^*(\alpha)$ and this choice of $\gamma^* = \gamma^*(\alpha)$ is the unique way to realize this.

4.2.4. Concluding remarks. As a conclusion, the condensed phase, where the partition function itself converges, deserves its own thermodynamics. It certainly has been unduly neglected in the literature concerning multifractals. Its outlines have been drawn in this section, taking advantage of the Gibbs framework of maximum likelihood estimation.

We now come to the related construction and questions of a singular measure presenting atoms and gaps.

4.3. Atoms: $r_{max,-} = 0$ and $\pi_{min,-} > 0$

As $r_{\max,-} := \max_{l=1,\dots,d} r_{l,-} \to 0$, while $\pi_{\min,-} := \min_{l=1,\dots,d} \pi_{l,-} > 0$, the sterile sub-intervals shrink to become points to which non-null sub-masses are affected. Measure μ will present atoms in the thermodynamic limit $n \to \infty$. In this context, $r_{l,-} = 0, l = 1, \dots, d$; hence, $\sum_{l=d+1}^{m} r_{l,+} = 1$ now expresses the volume conservation. As a result, $\tau(0) = -1$, in the previous construction. In contrast, the number q^+ still remains in the open interval $(0, 1)$. The formulae (4.1)–(4.4) only make sense for $q^* \geqslant 0$, in which case $z_-(q, q^*) = 0$, as a result of $0^{q^*} = 0$. The function $Z(s, q, q^*)$ defined in (4.2) becomes

$$Z(s, q, q^*) = \frac{1}{1 - b^s z_+(q, q^*)} \tag{4.19}$$

as $q^* \geqslant 0$ and $s \leqslant F(q, q^*) := \min(0, F_+(q, q^*))$. It diverges elsewhere.

Observe that $z_+(q, q^*) := \sum_{l=d+1}^{m} \pi_{l,+}^{q} r_{l,+}^{q^*}$ is now subject to the constraint $\sum_{l=d+1}^{m} r_{l,+} = 1$. Under this additional constraint, the function $\tau(q)$ remains implicitly defined by $F_+(q, -\tau(q)) = 0$, as in (4.4).

To take the atoms into account, the partition function (4.1) has to be replaced by its 'grid' version where 'space' is cut into equal pieces of length M^{-n} (see [9, p 56] for a similar account)

$$Z_n^G(q, q^*) := \sum_{i=1}^{M^n} \pi_{i,n}^{q} M^{-nq^*}.$$

In this formula, $\pi_{i,n}$ is the mass attributed to the ith sub-interval of length M^{-n} at resolution n. This results from the fact that the solution of (4.1) appears 'blind' to the atoms for which $z_-(q, q^*) = 0$.

If now $\tau_{a,n}(q)$ denotes the solution to

$$Z_n^G(q, -\tau_{a,n}(q)) = 1$$

we obtain, letting $\tau_a(q) = \lim_{n\to\infty} \tau_{a,n}(q)$,

$$\tau_a(q) = \lim_{n\to\infty} \log_{M^{-n}} \sum_{i=1}^{M^n} \pi_{i,n}^{q}.$$

In other words, the function $\tau(q)$, defined by $F_+(q, -\tau(q)) = 0$, has to be replaced by its 'atomic' *grid* version

$$\tau_a(q) := \tau(q) \qquad \text{if } \tau(q) \leqslant 0 \qquad \text{zero elsewhere} \tag{4.20}$$

with $\tau_a(0) = -1$ and $\tau_a(q^+) = 0$ for $0 < q^+ < 1$.

Define as before the Legendre transform $f(\alpha) := \inf_{q\in\mathbb{R}}(\alpha q - \tau(q))$ for this particular τ. It has the standard bell-shape, but its maximum is now $-\tau(0) = 1$. The Hausdorff spectrum of measure μ, with atoms, is now

$$f_{\mathrm{H}}(\alpha) = f(\alpha) \cdot \mathbf{1}\ (\alpha > 0) + 0 \cdot \mathbf{1}\ (\alpha = 0) \tag{4.21}$$

adding the point $(0, 0)$ to the graph of $f(\alpha)$: this reflects the fact that atoms are points where Hölder exponents are zero and that they form a set of (Lebesgue) measure zero.

Concerning the Legendre spectrum of measure μ, with atoms, it is

$$f_a(\alpha) := \inf_{q\in\mathbb{R}}(\alpha q - \tau_a(q)). \tag{4.22}$$

It has the form of the top and right portions of the previous graph of $f_{\rm H}$ down to $(\alpha_1, f_{\rm H}(\alpha_1))$, combined with a straight line joining this point to the point $(0, 0)$. Thus, for a singular measure μ with atoms $f_{\rm H} \neq f_a$ [9].

4.4. Gaps: $\pi_{max,-} = 0$ and $r_{min,-} > 0$

As $\pi_{\max,-} := \max_{l=1,\ldots,d}\pi_{l,-} \to 0$, the sterile sub-intervals, supported by non-null volumes this time, will receive no mass. Measure μ will present gaps. As $\pi_{l,-} = 0, l = 1, \ldots, d$, one gets $q^+ = 1$, since $\sum_{l=d+1}^{m}\pi_{l,+} = 1$ is the new equation of mass conservation. However, $0 > \tau(0) > -1$. The above construction (4.1)–(4.4) only makes sense for $q \geqslant 0$, in which case $z_-(q, q^*) = 0$. The function $Z(s, q, q^*)$ defined in (4.2) only makes sense in the restricted domain $q \geqslant 0$ and $s \leqslant F(q, q^*)$. The version of the function $\tau(q)$ defined by (4.4) with gaps is then

$$\tau_{\rm g}(q) := \tau(q) \qquad \text{if } q \geqslant 0 \qquad \text{zero otherwise} \tag{4.23}$$

with $0 > \tau_{\rm g}(0) > -1$ and $\tau_{\rm g}(1) = 1$.

Define as before the Legendre transform $f(\alpha) := \inf_{q\in\mathbb{R}}(\alpha q - \tau(q))$ for this particular τ. It has the standard bell-shape, with its maximum at $-\tau(0) < 1$, expressing the fact that gaps are present. The Hausdorff spectrum of measure μ, with atoms, is now

$$f_{\rm H}(\alpha) = f(\alpha)\cdot\mathbf{1}\ (\alpha < \infty) + 1\cdot\mathbf{1}\ (\alpha = \infty) \tag{4.24}$$

adding the point $(\infty, 1)$ to the graph of $f(\alpha)$. This reflects the fact that gaps are points where Hölder exponents are infinite and are of positive (Lebesgue) measure.

Concerning the Legendre spectrum of measure μ, with gaps it is

$$f_{\rm g}(\alpha) := \inf_{q\in\mathbb{R}}(\alpha q - \tau_{\rm g}(q)). \tag{4.25}$$

It has the form of the convex hull of $f_{\rm H}(\alpha)$: only the left-hand side of the graph of $f_{\rm H}$ remains identical, up to the point $(\alpha_0, -\tau(0))$, where $f_{\rm H}$ attains its maximum, $f_{\rm H}(\alpha_0) = -\tau(0) < 1$, combined with a straight line joining this point to point $(\infty, 1)$. Thus, for a singular measure μ with gaps $f_{\rm H} \neq f_{\rm g}$ [9].

Remark 3. Starting from $\tau_a(q)$, as defined by (4.20), observe that another function $\tau_a^*(q^*)$ could be defined by

$$-\tau_a^*(-\tau_a(q)) = q \qquad \text{and} \qquad -\tau_a(-\tau_a^*(q^*)) = q^*.$$

These functions are the inverses of one another. The graph of τ_a^* is of the type '$\tau_{\rm g}$' of some singular measure with gaps. As a result, a measure μ with atoms admits a reciprocal measure μ^* with gaps (and conversely), which is consistent with intuition.

Remark 4. Atoms, gaps and regular fragments may co-exist in the limit multifractal measure μ. To see this, partition the set of sterile indices $L_- := \{1, \ldots, d\}$ into three mutually disjoint sets (L_-^g, L_-^a, L_-^r) such that $L_- = L_-^g \cup L_-^a \cup L_-^r$. These sets are defined by the following properties

$$(\pi_{l,-} = 0, r_{l,-} > 0)_{l\in L_-^g} \qquad (\pi_{l,-} > 0, r_{l,-} = 0)_{l\in L_-^a} \qquad (\pi_{l,-} > 0, r_{l,-} > 0)_{l\in L_-^r}.$$

The version with atom gaps of the function $\tau(q)$ defined by (4.4) is then

$$\tau_{a,g}(q) := \tau(q) \qquad \text{if } q \geqslant 0 \text{ and } \tau(q) \leqslant 0 \qquad \text{zero otherwise.}$$

5. Towards more general self-similar bridges

5.1. Geometrical construction of self-similar bridges

The above constructions can be extended in the following way: as was underlined in the beginning of section 2, an equivalent way to look at the multifractal formalism of a probability measure on $[0, 1]$ is in terms of the distribution function at level n, say F_n, the graph of which is a path (or bridge) connecting $A := (0, 0)$ and $B := (1, 1)$. This path becomes very irregular as resolution n tends to infinity. However, in this context, F_n and its limit, F, as $n \to \infty$, was restricted to be a non-decreasing function of x (perhaps with gaps or jumps), which is a very particular way to join A and B. More general self-similar bridges from A to B can be constructed in the following way: initially, the connecting path looks like a straight line joining A and B.

At resolution $n = 1$, define recursively m points in the plane by

$$X_l = X_{l-1} + \Delta_l \qquad l = 1, \ldots, m \qquad X_0 = A \qquad X_m = B$$

with $\Delta_l := (\pi_l, r_l)$ where π_l and r_l are now non-null real numbers, standing respectively for the ordinate and abscissa of the increments. The condition $X_0 = A$, $X_m = B$ translates into $\sum_{l=1}^{m} \pi_l = 1$ and $\sum_{l=1}^{m} r_l = 1$. Draw a line joining X_{l-1} to $X_l, l = 1, \ldots, m$. This defines the (A, B)-path at resolution $n = 1$ as a continuous broken line made of m pieces. Suppose there are now two types of such m sublines: the 'sterile' ones $l = 1, \ldots, d < m-1$ whose increments will be specified to be $\Delta_{l,-} := (\pi_{l,-}, r_{l,-}) \in \mathbb{R}^2 - \{0, 0\}, l = 1, \ldots, d$, such that $\sum_{l=1}^{d} \pi_{l,-} := \pi_- \in \mathbb{R} - \{0\}$ and $\sum_{l=1}^{d} r_{l,-} := r_- \in \mathbb{R} - \{0\}$. These increments are sterile in the sense that the line joining X_{l-1} to X_l will remain unchanged in any subsequent step if $X_l - X_{l-1} = \Delta_{l,-}$ is a sterile increment. Concerning the $b := m - d > 1$ remaining 'productive' ones, their increments will be specified to be $\Delta_{l,+} := (\pi_{l,+}, r_{l,+}) \in \mathbb{R}^2 - \{0, 0\}, l = d+1, \ldots, m$, so that $\sum_{l=d+1}^{m} \pi_{l,+} := \pi_+ = 1 - \pi_- \in \mathbb{R}$ and $\sum_{l=d+1}^{m} r_{l,+} := r_+ = 1 - r_- \in \mathbb{R}$. For each such productive increment, define recursively m points in the plane by

$$X_{l,l'} = X_{l,l'-1} + \Delta_{l,+}^{l'} \qquad l' = 1, \ldots, m \qquad X_{l,0} = X_{l-1} \qquad X_{l,m} = X_l$$

with $\Delta_{l,+}^{l'} := (\pi_{l,+}\pi_{l'}, r_{l,+}r_{l'})$.

The (A, B)-path at resolution $n = 2$ is obtained while drawing a line joining $X_{l,l'-1}$ to $X_{l,l'}, l' = 1, \ldots, m$, as soon as l is such that $X_l - X_{l-1} = \Delta_{l,+}$ is a productive increment, leaving the sterile pieces unaffected. Upon indefinite iteration, we are also left with a singular self-similar (A, B)-path in dimension $k = 2$, the thermodynamic of which will be our objective.

We shall distinguish two cases depending on the modulus of the increments.

The confined case. Here $|\pi_l| < 1$ and $|r_l| < 1$ for all $l = 1, \ldots, m$, in which case the (A, B)-path is forced to stay within some bounded set as we now briefly show.

We shall prove this on the abscissa of the path, a similar argument holding for the ordinate. It is indeed possible to give a recurrent algorithm which yields, at resolution n, their maximal, say $X_{1,\max}(n)$, and minimal, say $X_{1,\min}(n)$, abscissa: indeed, at resolution $n-1$, $X_{1,\min}(n-1)$ and $X_{1,\max}(n-1)$ yield m local minima and maxima at step n, depending on the signs of $r_l, l = 1, \ldots, m$. These have to be compared in order to extract the desired

values at step n. This results from self-similarity. The obtained recurrence is

$$X_{1,\max}(n) = \max_{l=1,\dots,m} \left(\sum_{k=1}^{l-1} r_k + r_l (X_{1,\max}(n-1)\mathbf{1}_{(r_l>0)} + X_{1,\min}(n-1)\mathbf{1}_{(r_l<0)}) \right)$$

$$X_{1,\min}(n) = \min_{l=1,\dots,m} \left(\sum_{k=1}^{l-1} r_k + r_l (X_{1,\min}(n-1)\mathbf{1}_{(r_l>0)} + X_{1,\max}(n-1)\mathbf{1}_{(r_l<0)}) \right).$$

Moreover, $1 \leqslant X_{1,\max}(n)$ and $0 \geqslant X_{1,\min}(n)$.

Assuming $|r_l| < 1$ for all $l = 1, \dots, m$ yields a convergent algorithm. Indeed, indexing with a '+' the q values of the index l, say $l_1^+, \dots, l_q^+$, associated with a positive value of r_l and with a '−' the $m - q$ remaining ones, the above recurrent equations yield the finite value $X_{1,\max} := \lim_{n\to\infty} X_{1,\max}(n)$

$$X_{1,\max} = \max_{\substack{j'=1,\dots,m-q \\ j=1,\dots,m-q \\ i=1,\dots,q}} \left(\frac{\sum_{k=1}^{l_i^+-1} r_k}{1 - r_{l_i^+}}, \sum_{k=1}^{l_j^--1} r_k + r_{l_j^-} \frac{\sum_{k=1}^{l_i^+-1} r_k}{1 - r_{l_i^+}}, \frac{\sum_{k=1}^{l_j^--1} r_k + r_{l_j^-} \sum_{k=1}^{l_{j'}^--1} r_k}{1 - r_{l_j^-} r_{l_{j'}^-}} \right).$$

Concerning $X_{1,\min} := \lim_{n\to\infty} X_{1,\min}(n)$, it is the minimum over the same set of indices of the same values.

Thus, the (A, B)-path abscissa remains in the finite interval $[X_{1,\min}, X_{1,\max}]$. In a similar way, its ordinate remains in the finite interval $[X_{2,\min}, X_{2,\max}]$ substituting π_k to r_k in the above expressions, so that the (A, B)-path remains as a whole within $[X_{1,\min}, X_{1,\max}] \times [X_{2,\min}, X_{2,\max}]$.

The unconfined case. Here $|\pi_{l,+}| > 1$ or $|r_{l,+}| > 1$ for some $l = d+1, \dots, m$, in which case the (A, B)-path is not confined within any bounded region of the plane: 'giant' (i.e. singularly large) pieces will co-exist with 'tiny' (i.e. singularly small) and 'regular' pieces in the (A, B)-path. The prospective study of this case that follows will concentrate on the analysis of the sequence of curves generated as resolution increases.

Remark 5. These (A, B)-paths are fractal lines joining A and B, just like a Von Koch curve (for example) is one, but a Von Koch curve is only a monofractal which *cannot* be generated this way: an appeal to a geometrical representation of *substitutions* is necessary [27, 28].

Remark 6. These constructions can easily be extended to any Euclidean dimension k.

Remark 7. Jumps along the abscissa, ordinate and regular sublines may co-exist in the limit (A, B)-path. To do this, partition the set of sterile indices $L_- := \{1, \dots, d\}$ into three mutually disjoint sets (L_-^1, L_-^2, L_-^3) such that $L_- = L_-^1 \cup L_-^2 \cup L_-^3$. These sets are defined by the properties $(|\pi_{l,-}| = 0, |r_{l,-}| > 0)_{l\in L_-^1}$, $(|\pi_{l,-}| > 0, |r_{l,-}| = 0)_{l\in L_-^2}$ and $(|\pi_{l,-}| > 0, |r_{l,-}| > 0)_{l\in L_-^3}$, therefore allowing $(\pi_{l,-}, r_{l,-})$ to take the value zero in some places.

Remark 8. If $(r_l)_{l=1,\dots,m} \in (0, 1)$ while $(\pi_l)_{l=1,\dots,m} \in \mathbb{R} - \{0\}$ in the above construction, an (A, B)-path is still the distribution function of some signed real-valued Borel measure μ supported by the interval $[0, 1]$. The distribution function $x \in [0, 1] \to F(x) := \mu([0, x]) \in \mathbb{R}$ is only right-continuous with $\mu([0, 1]) = 1$, but remains self-similar. We thus relaxed the conditions that μ is $[0, 1]$-valued and that $F(x)$ should be non-decreasing, while maintaining the other constraints: this represents, for example, the distribution of a unit *charge* on the unit interval $[0, 1]$.

Thus, self-affine functions are special cases of bridges as introduced in this section. They have also been studied widely, especially from a multifractal point of view by Jaffard [29], who showed that, in general, the multifractal formalism for functions only yields an upper bound for the singularity spectrum, whereas it is exact for self-similar ones (with a definition of 'self-similarity' which does not exactly fit with ours (see our conclusion in section 6)). Also, Falconer and O'Neil [30] have taken into account vector-valued multifractal measures, a special case of which are the self-affine functions just mentioned.

If $(\pi_l)_{l=1,\dots,m} \in (0,1)$ while $(r_l)_{l=1,\dots,m} \in \mathbb{R} - \{0\}$ in the above construction, an (A,B)-path represents a physical phenomenon where a volume contraction is permitted while adding mass (or even charge if $\pi_l < 0$ for some l) within the system.

5.2. Statistics of the (A,B)-path trail

5.2.1. Partition function analysis. We now come to the extension of the thermodynamic formalism to such (A,B)-paths, which amounts to the parametrization of the abscissa of the (A,B)-path by its ordinate (a reciprocal problem would of course consist in a parametrization of the ordinate by the abscissa).

Let $\pi_{l,\pm} = |\pi_{l,\pm}|\,\mathrm{e}^{-\mathrm{i}\pi b_l(\pi_{l,\pm})}$, $l = 1,\dots,m$, be the modulus–phase representation of $\pi_{l,\pm}$, with $b_l(\pi_{l,\pm}) = 0$ if $\pi_{l,\pm} > 0$, and $b_l(\pi_{l,\pm}) = 1$ if $\pi_{l,\pm} < 0$, controlling the sign of $\pi_{l,\pm} := (\pi_{l,+}, \pi_{l,-})$. (Note that $b_l(\pi_{l,\pm}) = \frac{1}{2}(1 - \mathrm{sign}(\pi_{l,\pm}))$.) In a similar way, let $r_{l,\pm} = |r_{l,\pm}|\,\mathrm{e}^{-\mathrm{i}\pi b_l(r_{l,\pm})}$, $l = 1,\dots,m$, be the modulus–phase representation of $r_{l,\pm} := (r_{l,+}, r_{l,-})$.

For $(\boldsymbol{q} := (q_1, q_2), \boldsymbol{q}^* := (q_1^*, q_2^*)) \in \mathbb{R}^4$, define the extended level-n partition function

$$Z_n(\boldsymbol{q}, \boldsymbol{q}^*) := \sum_{i=1}^{N_n} |\pi_i(n)|^{q_1}\,\mathrm{e}^{-q_2 b(\pi_i(n))} |r_i(n)|^{q_1^*}\,\mathrm{e}^{-q_2^* b(r_i(n))} \qquad Z_n(\mathbf{0},\mathbf{0}) = N_n \tag{5.1}$$

where $(|\pi_i(n)|, b(\pi_i(n)); |r_i(n)|, b(r_i(n)))_{i=1,\dots,N_n}$ is the vector of increments' representation attached to each of the N_n available (A,B)-path pieces at resolution n.

Function $Z_n(\boldsymbol{q}, \boldsymbol{q}^*)$ is now defined recursively by

$$Z_{n+1}(\boldsymbol{q}, \boldsymbol{q}^*) = z_-(\boldsymbol{q}, \boldsymbol{q}^*) + Z_n(\boldsymbol{q}, \boldsymbol{q}^*) z_+(\boldsymbol{q}, \boldsymbol{q}^*) \qquad Z_0(\boldsymbol{q}, \boldsymbol{q}^*) = 1 \tag{5.2}$$

with

$$z_-(\boldsymbol{q}, \boldsymbol{q}^*) := \sum_{l=1}^{d} |\pi_{l,-}|^{q_1}\,\mathrm{e}^{-q_2 b_l(\pi_{l,-})} |r_{l,-}|^{q_1^*}\,\mathrm{e}^{-q_2^* b_l(r_{l,-})}$$

and

$$z_+(\boldsymbol{q}, \boldsymbol{q}^*) := \sum_{l=d+1}^{m} |\pi_{l,+}|^{q_1}\,\mathrm{e}^{-q_2 b_l(\pi_{l,+})} |r_{l,+}|^{q_1^*}\,\mathrm{e}^{-q_2^* b_l(r_{l,+})}.$$

Next define the generating function $Z(s, \boldsymbol{q}, \boldsymbol{q}^*) := \sum_{n \geqslant 0} b^{sn} Z_n(\boldsymbol{q}, \boldsymbol{q}^*)$. It follows from (5.2) that

$$Z(s, \boldsymbol{q}, \boldsymbol{q}^*) = \frac{1 - b^s(1 - z_-(\boldsymbol{q}, \boldsymbol{q}^*))}{(1 - b^s)(1 - b^s z_+(\boldsymbol{q}, \boldsymbol{q}^*))} \tag{5.3}$$

provided $s \leqslant F(\boldsymbol{q}, \boldsymbol{q}^*) := \min(0, F_+(\boldsymbol{q}, \boldsymbol{q}^*))$, with $F_+(\boldsymbol{q}, \boldsymbol{q}^*) := -\log_b z_+(\boldsymbol{q}, \boldsymbol{q}^*)$, and is undefined otherwise.

The function $F_+ : \mathbb{R}^4 \to \mathbb{R}$ is concave and analytic, satisfying $F_+(\mathbf{0},\mathbf{0}) = -1$. The function F is also concave but only continuous, and this is the signature of a *phase transition* while crossing the critical domain: $\{(\boldsymbol{q}, \boldsymbol{q}^*) \in \mathbb{R}^4 : F_+(\boldsymbol{q}, \boldsymbol{q}^*) = 0\}$.

It now follows from (5.3) that

$$\lim_{n\to\infty} -\log_{N_n} Z_n(\boldsymbol{q}, \boldsymbol{q}^*) = F(\boldsymbol{q}, \boldsymbol{q}^*). \tag{5.4}$$

Depending on whether one is in a confined or unconfined situation, the analysis may vary.

5.2.2. The confined case. Suppose first that the (A, B)-path is confined. We wish here to derive the thermodynamics of the modulus–phase information on the abscissa against the modulus-phase information on the ordinate (see the definitions of the sets $S_1(\alpha;\gamma^*)$ and $S_2(\alpha';\gamma^*)$, later).

The critical domain equations. Define then the two functions $\tau(\boldsymbol{q}) := (\tau_1(\boldsymbol{q}), \tau_2(\boldsymbol{q}))$ implicitly by

$$F_+(\boldsymbol{q}, (-\tau_1(\boldsymbol{q}), 0)) = 0 \tag{5.5}$$

and

$$F_+(\boldsymbol{q}, (0, -\tau_2(\boldsymbol{q}))) = 0. \tag{5.6}$$

These functions are particular cuts $(\tau_1(\boldsymbol{q}) := \widetilde{\tau}_1(\boldsymbol{q}, 0), \tau_2(\boldsymbol{q}) := \widetilde{\tau}_2(0, \boldsymbol{q}))$ of the two functions $(\widetilde{\tau}_1(\boldsymbol{q}, q_2^*), \widetilde{\tau}_2(q_1^*, \boldsymbol{q}))$ defined respectively by

$$F_+(\boldsymbol{q}, (-\widetilde{\tau}_1(\boldsymbol{q}, q_2^*), q_2^*)) = 0 \qquad \text{and} \qquad F_+(\boldsymbol{q}, (q_1^*, -\widetilde{\tau}_2(q_1^*, \boldsymbol{q}))) = 0.$$

The equations $q_1^* = \widetilde{\tau}_1(\boldsymbol{q}, q_2^*)$ and $q_2^* = \widetilde{\tau}_2(q_1^*, \boldsymbol{q})$ are two alternative descriptions of the critical domain $\{(\boldsymbol{q}, \boldsymbol{q}^*) \in \mathbb{R}^4 : F_+(\boldsymbol{q}, \boldsymbol{q}^*) = 0\}$.

Each deserves a particular study.

• Concerning the function $\tau_1 : \mathbb{R}^2 \to \mathbb{R}$, it is concave and analytic, and is defined by $\sum_{l=d+1}^m |\pi_{l,+}|^{q_1} \mathrm{e}^{-q_2 b_l(\pi_{l,+})} |r_{l,+}|^{-\tau_1(\boldsymbol{q})} = 1$. At $\boldsymbol{q} = \boldsymbol{0}$, it takes the value $\tau_1(\boldsymbol{0})$ defined uniquely by $\sum_{l=d+1}^m |r_{l,+}|^{-\tau_1(\boldsymbol{0})} = 1$, with $\tau_1(\boldsymbol{0}) < -1$ if and only if $\sum_{l=d+1}^m |r_{l,+}| > 1$.

• Concerning the function τ_2, implicitly defined by

$$\sum_{l=d+1}^m |\pi_{l,+}|^{q_1} \mathrm{e}^{-q_2 b_l(\pi_{l,+})} \mathrm{e}^{\tau_2(\boldsymbol{q}) b_l(r_{l,+})} = 1$$

we have

$$\tau_2(\boldsymbol{q}) = -\log\left(\frac{z_{+-}(q_1) + z_{--}(q_1)\,\mathrm{e}^{-q_2}}{1 - (z_{++}(q_1) + z_{-+}(q_1)\,\mathrm{e}^{-q_2})}\right)$$

where $(z_{++}, z_{+-}, z_{-+}, z_{--})$ are defined by

$$z_+(\boldsymbol{q}, (0, q_2^*)) = z_{++}(q_1) + z_{+-}(q_1)\,\mathrm{e}^{-q_2^*} + z_{-+}(q_1)\,\mathrm{e}^{-q_2} + z_{--}(q_1)\,\mathrm{e}^{-q_2}\,\mathrm{e}^{-q_2^*}$$

upon specifying the joint positiveness (or not) of $(\pi_{l,+}, r_{l,+})_{l=d+1,\ldots,m}$.

Hence, the function τ_2 is concave and analytic, on the convex domain defined by

$$\{(q_1, q_2) \in \mathbb{R}^2 : (z_{--}(q_1) + z_{+-}(q_1)\,\mathrm{e}^{-q_2}) < 1\}$$

not including zero.

The Legendre transforms of $\tau_1(\boldsymbol{q})$, $\tau_2(\boldsymbol{q})$. Define next their Legendre transforms $(f_1(\alpha), f_2(\alpha'))$ with

$$f_1(\alpha) := \inf_{\boldsymbol{q}\in\mathbb{R}^2} (\langle \alpha, \boldsymbol{q}\rangle - \tau_1(\boldsymbol{q})) \tag{5.7}$$

$$f_2(\alpha') := \inf_{\boldsymbol{q}\in\mathbb{R}^2} (\langle \alpha', \boldsymbol{q}\rangle - \tau_2(\boldsymbol{q})). \tag{5.8}$$

We are then led to the idea that a bispectrum should be meaningful when parametrizing the abscissa of a self-similar (A, B)-path by its ordinate: this is a result of the reduction factors being allowed to be negative, and hence encoded by both their moduli and signs.

Concerning the Legendre transform $f_1(\alpha)$, its dual system of Legendre variables $\alpha := (\alpha_1, \alpha_2)$ is defined on its (bounded) Hölder range which is the polygonal convex hull, say H, of the points $(-\log|\pi_{l,+}|/-\log|r_{l,+}|, b(\pi_{l,+})/-\log|r_{l,+}|)_{l=d+1,\ldots,m}$.

Concerning the Legendre transform $f_2(\alpha')$, its dual system of Legendre variables $\alpha' := (\alpha'_1, \alpha'_2)$ is defined on its (unbounded) Hölder range which is the polygonal convex hull, say H', of the points $(-\log|\pi_{l,+}|/b(r_{l,+}), b(\pi_{l,+})/b(r_{l,+}))_{l=d+1,\ldots,m}$. This domain is included in the unbounded domain $[\min_l(-\log|\pi_{l,+}|/b(r_{l,+})), \infty]\times[0, \infty]$, upon projecting.

We shall call these functions the Legendre spectra of the self-similar (A, B)-path: each is non-negative, concave and analytic on its definition domain. Function f_1 attains its maximum at $\alpha_0 := (\alpha_{1,0}, \alpha_{2,0})$ which is the centre of gravity of the extremal points of H. Moreover, $f_1(\alpha_0) = -\tau_1(\mathbf{0})$. Concerning the function f_2, the centre of gravity of the extremal points of H', say $\alpha'_0 := (\alpha'_{1,0}, \alpha'_{2,0})$, is rejected at infinity.

The functions $(f_1(\alpha), f_2(\alpha))$ are identified with the Hausdorff spectra of the self-similar (A, B)-paths just constructed. We now come to the interpretation of these functions.

The thermodynamical interpretation. Define the set $S_1(\alpha;\gamma^*)$ by

$$\left\{ i \in [N_n] : \frac{-\log|\pi_i(n)|}{-\log|r_i(n)|} \to \alpha_1, \frac{b(\pi_i(n))}{-\log|r_i(n)|} \to \alpha_2; -\log_{N_n}|r_i(n)| \to \gamma^* \right\}$$

and let $N_n^1(\alpha;\gamma^*) := \#\{S_1(\alpha;\gamma^*)\}$ be the cardinal of this set. It is the number of atoms whose first Hölder exponent is α *and* for which $|r_i(n)| \sim N_n^{-\gamma^*}(\gamma^* > 0)$.

In a similar way, define the set $S_2(\alpha';\gamma^*)$ by

$$\left\{ i \in [N_n] : \frac{-\log|\pi_i(n)|}{b(r_i(n))} \to \alpha'_1, \frac{b(\pi_i(n))}{b(r_i(n))} \to \alpha'_2; \frac{b(r_i(n))}{\log N_n} \to \gamma^* \right\}$$

and $N_n^2(\alpha';\gamma^*) := \#\{S_2(\alpha';\gamma^*)\}$ its cardinal. It is the number of atoms whose second Hölder exponent is α' *and* for which $b(r_i(n)) \sim \log N_n^{\gamma^*}$. We learn from large deviation theory that

$$\lim_{n\to\infty} \log_{N_n} N_n^1(\alpha;\gamma^*) = f_\alpha^1(\gamma^*) \tag{5.9}$$

$$\lim_{n\to\infty} \log_{N_n} N_n^2(\alpha';\gamma^*) = f_{\alpha'}^2(\gamma^*) \tag{5.10}$$

where $f_\alpha^1(\gamma^*)$ (respectively, $f_{\alpha'}^2(\gamma^*)$) is the α-cut (respectively, α'-cut):

$$f_\alpha^1(\gamma^*) = f^1(\alpha_1\gamma^*, \alpha_2\gamma^*, \gamma^*) \qquad \text{(respectively, } f_{\alpha'}^2(\gamma^*) = f^2(\alpha'_1\gamma^*, \alpha'_2\gamma^*, \gamma^*))$$

with

$$f^1(\gamma_1, \gamma_2, \gamma^*) := \inf_{(\boldsymbol{q},q^*)\in\mathbb{R}^3} (\langle\gamma, \boldsymbol{q}\rangle - F(\boldsymbol{q}, (q_1^*, 0)))$$

$$\left(\text{respectively, } f^2(\gamma_1, \gamma_2, \gamma^*) := \inf_{(\boldsymbol{q},q^*)\in\mathbb{R}^3} (\langle\gamma, \boldsymbol{q}\rangle - F(\boldsymbol{q}, (0, q_2^*)))\right).$$

The function $f_\alpha^1(\gamma^*)$ possesses the following properties. For any $\alpha \in H$, it is positive concave as a function of γ^*. It takes the maximal value $\sigma(\alpha) := f_\alpha^1(0, 0)$. There exists a unique line passing through the origin which is tangent to the γ^*-graph of $f_\alpha^1(\gamma^*)$ and whose slope is $f_1(\alpha)$. This line hits this graph at $\gamma^* := \gamma_1^*(\alpha)$. Hence, $f_1(\alpha) = f_\alpha^1(\gamma_1^*(\alpha))/\gamma_1^*(\alpha)$.

Defining $r_n^1(\alpha) := N_n^{-\gamma_1^*(\alpha)}$, we get as in section 3

$$\lim_{n\to\infty} \log_{1/r_n^1(\alpha)} N_n^1(\alpha; \gamma_1^*(\alpha)) = f_1(\alpha). \tag{5.11}$$

Working in a similar way on $f^2_{\alpha'}(\gamma^*)$, we get with $r_n^2(\alpha') := N_n^{-\gamma_2^*(\alpha')}$ and $f_2(\alpha') = f^2_{\alpha'}(\gamma_2^*(\alpha'))/\gamma_2^*(\alpha')$

$$\lim_{n\to\infty} \log_{1/r_n^2(\alpha')} N_n^2(\alpha'; \gamma_2^*(\alpha')) = f_2(\alpha'). \tag{5.12}$$

Concluding remarks. In this first extension, confined (A, B)-paths are the outcome of a deterministic conservative multiplicative cascade whose reduction factors are allowed to be negative, but less than one in modulus. As a result, the partition function should enclose information on both their moduli and signs: an immediate analysis extending that of multifractal measures holds.

5.2.3. The unconfined case. Suppose now that the (A, B)-path is unconfined. We shall limit ourselves to the particular case $|r_{\max,+}| := \max_{l=d+1,\dots,m} |r_{l,+}| > 1$ and $|r_{\min,+}| := \min_{l=d+1,\dots,m} |r_{l,+}| < 1$, while $|\pi_{l,+}| < 1$, $l = d+1, \dots, m$ (a complete study of all the situations that can arise is left to a future work).

As was noted previously, giant fragments co-exist with tiny ones in the (A, B)-path, and one expects the previous analysis to change in a drastic way. This is what happens, and it affects the function τ_1.

Equation (5.5) that defines the function τ_1 takes the equivalent form

$$z_+(\boldsymbol{q}, (-\tau_1(\boldsymbol{q}), 0)) = 1$$

with $z_+(\boldsymbol{q}, q_1^*) := z_+(\boldsymbol{q}, (q_1^*, 0)) = \sum_{l=d+1}^m |\pi_{l,+}|^{q_1} \mathrm{e}^{-q_2 b_l(\pi_{l,+})} |r_{l,+}|^{q_1^*}$.

This function, as a function of q_1^*, is now such that $z_+(\boldsymbol{q}, q_1^*) \underset{q_1^*\to\pm\infty}{\to} +\infty$, as a result of $|r_{\max,+}| > 1$ and $|r_{\min,+}| < 1$. It now has a unique absolute minimum $q_1^*(\boldsymbol{q})$ defined by

$$\partial_{q_1^*} z_+(\boldsymbol{q}, q_1^*(\boldsymbol{q})) = 0.$$

Moreover, $z_+(\boldsymbol{q}, q_1^*(\boldsymbol{q})) > 0$. Thus equation (5.5) now has none, one or two solutions, depending on whether $z_+(\boldsymbol{q}, q_1^*(\boldsymbol{q})) > 1, = 1$, or <1, respectively: the unique solution now *bifurcates* into two solutions.

Suppose $z_+(\boldsymbol{q}, q_1^*(\boldsymbol{q})) < 1$, so that equation (5.5) has two solutions there. This defines a convex domain, say C, in the $\boldsymbol{q}$-plane, not including $\boldsymbol{q} = \boldsymbol{0}$. In this domain, we let $\tau_1^s(\boldsymbol{q})$, $\tau_1^l(\boldsymbol{q})$ denote these two solutions, with of course

$$\tau_1^s(\boldsymbol{q}) = \tau_1^l(\boldsymbol{q}) \qquad \text{if } \boldsymbol{q} \in \partial C := \{\boldsymbol{q} \in \mathbb{R}^2 : z_+(\boldsymbol{q}, q_1^*(\boldsymbol{q})) = 1\}$$

which shows that the two solutions merge at the boundary ∂C of the domain C. Moreover, $|\nabla \tau_1^{s,l}(\boldsymbol{q})| \to +\infty$ as $\boldsymbol{q}$ approaches ∂C.

The function $\tau_1^s(\boldsymbol{q}) : C \to \mathbb{R}$ is monotone increasing and concave while $\tau_1^l(\boldsymbol{q}) : C \to \mathbb{R}$ is monotone decreasing and convex. The function $\tau_1^s(\boldsymbol{q})$ is representative of the tiny (small) fragments constituting the (A, B)-path, for which $|r_i(n)| \underset{n\to\infty}{\to} 0$, whereas $\tau_1^l(\boldsymbol{q})$ is representative of its giant (large) pieces, for which $|r_i(n)| \underset{n\to\infty}{\to} +\infty$. Considering their Legendre transforms $(f_1^s(\alpha), f_1^l(\alpha))$ with

$$f_1^{s,l}(\alpha) := \inf_{\boldsymbol{q}\in C} (\langle \alpha, \boldsymbol{q}\rangle - \tau_1^{s,l}(\boldsymbol{q}))$$

we have:

• $f_1^s(\alpha) \geqslant 0$ with $\alpha := (\alpha_1 > 0, \alpha_2 > 0)$ defined on the induced (unbounded) Hölder range;

• $f_1^l(\alpha) \leqslant 0$ with $\alpha := (\alpha_1 < 0, \alpha_2 < 0)$ defined on the induced (unbounded) Hölder range.

We now come to the interpretation of these functions $(f_1^s(\alpha), f_1^l(\alpha))$. Remember that $N_n^1(\alpha; \gamma^*) := \#\{S_1(\alpha; \gamma^*)\}$ is the number of atoms whose first Hölder exponent is α *and* for which $|r_i(n)| \sim N_n^{-\gamma^*}$. Two cases may now arise depending on the type of fragments of interest (tiny or giant, respectively).

• $\gamma^* > 0$, so that $|r_i(n)| \underset{n\to\infty}{\to} 0$: let $N_n^{1,s}(\alpha; \gamma^*)$ denote the cardinal of the set $S_1(\alpha; \gamma^*)$ when γ^* is forced to be positive. Following the above analysis, there exists a unique $\gamma_s^*(\alpha) > 0$, such that, if $r_n^s(\alpha) := N_n^{-\gamma_s^*(\alpha)}$, then

$$\lim_{n\to\infty} \log_{1/r_n^s(\alpha)} N_n^{1,s}(\alpha; \gamma_s^*(\alpha)) = f_1^s(\alpha) \geqslant 0$$

extending (5.11).

• $\gamma^* < 0$, so that $|r_i(n)| \underset{n\to\infty}{\to} +\infty$: if $N_n^{1,l}(\alpha; \gamma^*)$ denotes the cardinal of the set $S_1(\alpha; \gamma^*)$ when γ^* is forced to be negative, there exists a unique $\gamma_l^*(\alpha) < 0$, such that, if $r_n^l(\alpha) := N_n^{-\gamma_l^*(\alpha)}$, then

$$\lim_{n\to\infty} \log_{1/r_n^l(\alpha)} N_n^{1,l}(\alpha; \gamma_l^*(\alpha)) = f_1^l(\alpha) \leqslant 0$$

which is the version of (5.11) for the very large pieces constituting the (A, B)-path.

Example. Let $d = 0$, $m = 2$, $\pi_{1,+} = \pi_{2,+} = 1/2$ and $r_{1,+} = \alpha$ (the golden number $(1 + \sqrt{5})/2$), $r_{2,+} = -1/\alpha$.

Dropping the information on sign ($q_2 = 0$), we get

$$z_1(q_1; q_1^*) = (\tfrac{1}{2})^{q_1}(\alpha^{q_1^*} + \alpha^{-q_1^*}).$$

The unique minimum $q_1^*(q_1)$ is defined by $\alpha^{q_1^*(q_1)} = \alpha^{-q_1^*(q_1)}(\partial_{q_1^*} z_1 = 0)$, and hence $q_1^*(q_1) = 0$ and $z_1(q_1; q_1^*(q_1) = 0) := (\frac{1}{2})^{q_1 - 1}$ has to be compared to 1. If $q_1 < 1$ there is no solution to the degree-two equation $z_1(q_1; q_1^*) = 1$, if $q_1 = 1$ there is one solution to $z_1 = 1$, while $q_1 > 1$ yields the two solutions

$$q_1^* := -\tau_1^l(q_1) = \log_\alpha(2^{q_1 - 1} + \sqrt{2^{2(q_1-1)} - 1})$$
$$q_1^* := -\tau_1^s(q_1) = \log_\alpha(2^{q_1 - 1} - \sqrt{2^{2(q_1-1)} - 1}).$$

The Legendre transform $f_1^s(\alpha_1)$ of $\tau_1^s(q_1)$ is positive with support $[\log_\alpha 2, +\infty)$. It is increasing and diverges at $\alpha_1 = \infty$. Concerning the Legendre transform $f_1^l(\alpha_1)$ of $\tau_1^l(q_1)$, it is negative with support $(-\infty, -\log_\alpha 2]$. This function decreases and diverges at $\alpha_1 = -\infty$.

5.3. Statistics of the (A, B)-path graph

5.3.1. Spacetime partition function. In the above construction, we were interested in the statistics of an (A, B)-path trail, when the abscissa is parametrized by the ordinate. Adding one dimension to this problem amounts to developing this trail in time. So doing, we gain some insight into the (A, B)-path graph; some walker is now assumed to take a unit time to reach B, starting from A. An (A, B)-path graph is thus an (A', B')-path trail where $A' := (0, 0, 0)$, $B' := (1, 1, 1)$ with the first two components relative to space (abscissa and ordinate if the dimension of the underlying Euclidean space is $k = 2$) and the third component relative to (non-decreasing) time.

For $(\boldsymbol{q} := (q_1, q_2), \boldsymbol{q}^* := (q_1^*, q_2^*), p) \in \mathbb{R}^{2k+1}$, define the extended spacetime level-n partition function

$$Z_n(\boldsymbol{q}, \boldsymbol{q}^*, p) := \sum_{i=1}^{N_n} |\pi_i(n)|^{q_1} \mathrm{e}^{-q_2 b(\pi_i(n))} |r_i(n)|^{q_1^*} \mathrm{e}^{-q_2^* b(r_i(n))} t_i(n)^p \qquad Z_n(\mathbf{0}, \mathbf{0}, 0) = N_n$$

where $(|\pi_i(n)|, b(\pi_i(n)); |r_i(n)|, b(r_i(n)); t_i(n))_{i=1,\dots,N_n}$ is the vector of increments attached to each of the N_n available (A, B)-path graph pieces at resolution n.

We assume, with little loss of generality that $t_i(n) = N_n^{-1}$: the fraction of time spent by the walker on chunk $i \in [N_n]$ is identical for any chunk. A model for which the $(0, 1)$-valued fraction of time (say $(t_{l,-})_{l=1,\dots,d}$, $(t_{l,+})_{l=d+1,\dots,m}$) spent by the walker depends on the type of chunk under consideration is easy to imagine and to handle: the walker would trigger a clock specific to each type of chunk he experiences. It is, therefore, assumed in the following that $t_{l,-} = t_{l,+} = m^{-1}$ which is consistent with $\sum_{l=1,\dots,d} t_{l,-} + \sum_{l=d+1,\dots,m} t_{l,+} = 1$.

Now $Z_n(\boldsymbol{q}, \boldsymbol{q}^*, p)$ is defined recursively by

$$Z_{n+1}(\boldsymbol{q}, \boldsymbol{q}^*, p) = z_-(\boldsymbol{q}, \boldsymbol{q}^*, p) + Z_n(\boldsymbol{q}, \boldsymbol{q}^*, p) z_+(\boldsymbol{q}, \boldsymbol{q}^*, p), \;\; Z_0(\boldsymbol{q}, \boldsymbol{q}^*, p) = 1 \tag{5.13}$$

with

$$z_-(\boldsymbol{q}, \boldsymbol{q}^*, p) := \sum_{l=1}^{d} |\pi_{l,-}|^{q_1} \mathrm{e}^{-q_2 b_l(\pi_{l,-})} |r_{l,-}|^{q_1^*} \mathrm{e}^{-q_2^* b_l(r_{l,-})} t_{l,-}^p$$

and

$$z_+(\boldsymbol{q}, \boldsymbol{q}^*, p) := \sum_{l=d+1}^{m} |\pi_{l,+}|^{q_1} \mathrm{e}^{-q_2 b_l(\pi_{l,+})} |r_{l,+}|^{q_1^*} \mathrm{e}^{-q_2^* b_l(r_{l,+})} t_{l,+}^p.$$

Here $t_{l,-} = t_{l,+} = m^{-1}$, under our hypothesis.

Define the generating function

$$Z(s, \boldsymbol{q}, \boldsymbol{q}^*, p) := \sum_{n \geqslant 0} b^{sn} Z_n(\boldsymbol{q}, \boldsymbol{q}^*, p).$$

It follows from (5.13) that

$$Z(s, \boldsymbol{q}, \boldsymbol{q}^*, p) = \frac{1 - b^s(1 - z_-(\boldsymbol{q}, \boldsymbol{q}^*, p))}{(1 - b^s)(1 - b^s z_+(\boldsymbol{q}, \boldsymbol{q}^*, p))} \tag{5.14}$$

provided $s \leqslant F(\boldsymbol{q}, \boldsymbol{q}^*, p) := \min(0, F_+(\boldsymbol{q}, \boldsymbol{q}^*, p))$, with

$$F_+(\boldsymbol{q}, \boldsymbol{q}^*, p) := -\log_b z_+(\boldsymbol{q}, \boldsymbol{q}^*, p)$$

the function z is undefined otherwise.

5.3.2. The equation of the spacetime critical domain. One wishes here to study the asymptotic behaviour of the (A, B)-path pieces borrowed by the walker which reduces to the understanding of the set $B(\alpha; \alpha^*)$ defined in subsection 5.3.4.

The function $F_+ : \mathbb{R}^5 \to \mathbb{R}$ is concave and analytic, satisfying $F_+(\mathbf{0}, \mathbf{0}, 0) = -1$, and $F_+(\mathbf{0}, \mathbf{0}, \log_m b) = 0$. The function F is also concave but only continuous, and we have a *phase transition* while crossing the critical domain:

$$\{(\boldsymbol{q}, \boldsymbol{q}^*, p) \in \mathbb{R}^5 : F_+(\boldsymbol{q}, \boldsymbol{q}^*, p) = 0\}.$$

It is now a consequence of (5.14) that

$$\lim_{n \to \infty} -\log_{N_n} Z_n(\boldsymbol{q}, \boldsymbol{q}^*, p) = F(\boldsymbol{q}, \boldsymbol{q}^*, p). \tag{5.15}$$

Now define the function $\tau(\boldsymbol{q}, \boldsymbol{q}^*)$ implicitly by

$$F_+(\boldsymbol{q}, \boldsymbol{q}^*, -\tau(\boldsymbol{q}, \boldsymbol{q}^*)) = 0.$$

In developed form

$$z_+(\boldsymbol{q}, \boldsymbol{q}^*, -\tau(\boldsymbol{q}, \boldsymbol{q}^*)) = \sum_{l=d+1}^{m} |\pi_{l,+}|^{q_1} \mathrm{e}^{-q_2 b_l(\pi_{l,+})} |r_{l,+}|^{q_1^*} \mathrm{e}^{-q_2^* b_l(r_{l,+})} t_{l,+}^{-\tau(\boldsymbol{q},\boldsymbol{q}^*)} = 1. \tag{5.16}$$

It has (under our simplifying assumptions $t_{l,-} = t_{l,+} = m^{-1}$) the explicit expression

$$\tau(\boldsymbol{q}, \boldsymbol{q}^*) = -\log_m z_+(\boldsymbol{q}, \boldsymbol{q}^*)$$

with $z_+(\boldsymbol{q}, \boldsymbol{q}^*) := z_+(\boldsymbol{q}, \boldsymbol{q}^*, 0)$.

The function $\tau : \mathbb{R}^4 \to \mathbb{R}$ is concave and analytic. At $(\boldsymbol{q}, \boldsymbol{q}^*) = (\mathbf{o}, 0)$, it takes the value $\tau(\mathbf{o}, \mathbf{o}) = -\log_m b > -1$.

5.3.3. The Legendre transform. Define next its Legendre transform $f(\alpha, \alpha^*)$ by

$$f(\alpha, \alpha^*) := \inf_{(\boldsymbol{q},\boldsymbol{q}^*)\in\mathbb{R}^4} (\langle \alpha, \boldsymbol{q} \rangle + \langle \alpha^*, \boldsymbol{q}^* \rangle - \tau(\boldsymbol{q}, \boldsymbol{q}^*)). \tag{5.17}$$

Its dual system of Legendre variables $\alpha := (\alpha_1, \alpha_2)$ and $\alpha^* := (\alpha_1^*, \alpha_2^*)$ is defined on its (bounded) Hölder range which is the polygonal convex hull, say H, of the points $(-\log_m |\pi_{l,+}|, b(\pi_{l,+})/\log m, -\log_m |r_{l,+}|, b(r_{l,+})/\log m)_{l=d+1,\ldots,m}$.

We shall call this function the Legendre spectrum of the self-similar (A, B)-path graph: it is non-negative, concave and analytic on its definition domain. The function f attains its maximum at (α_0, α_0^*) which is the centre of gravity of the extremal points of H. Moreover, $f(\alpha_0, \alpha_0^*) = -\tau(\mathbf{o}, \mathbf{o}) < 1$, expressing the fact that macroscopic fragments are present within such (A, B)-paths.

5.3.4. The thermodynamical interpretation. We now come to the interpretation of this function.

Define the set $B(\alpha; \alpha^*)$ by

$$\left\{ i \in [N_n] : \frac{-\log |\pi_i(n)|}{\log N_n} \to \alpha_1, \frac{b(\pi_i(n))}{\log N_n} \to \alpha_2; \frac{-\log |r_i(n)|}{\log N_n} \to \alpha_1^*, \frac{b(r_i(n))}{\log N_n} \to \alpha_2^* \right\}$$

and let $N_n(\alpha; \alpha^*) := \#B(\alpha; \alpha^*)$ be the cardinal of this set. It is the number of atoms in the (A, B)-path graph whose Hölder exponent is (α, α^*). Note that the quantity $\log N_n$ normalizing the spatial information in the above set S is nothing but $-\log t_i(n)$, under our hypothesis that $t_i(n) = N_n^{-1}$.

Large deviation theory tells us that

$$\lim_{n\to\infty} \log_{N_n} N_n(\alpha; \alpha^*) = f(\alpha, \alpha^*). \tag{5.18}$$

In this context, the function $f(\alpha, \alpha^*)$ is also of some importance, since it coincides with the Hausdorff spectrum of the singular part of the self-similar (A, B)-path graph just constructed; in other words, $f(\alpha, \alpha^*) = f_{\mathrm{H}}(\alpha, \alpha^*)$ is the Hausdorff dimension of the set $B(\alpha; \alpha^*)$.

(The proof of this assertion can also be derived while readily adapting [10, theorem 16, p 13], transferring results from the code space to the embedding geometrical space.)

Remark 9. Of course, it is still possible to focus on the regular part of the (A, B)-path graph. This analysis would follow that explained in section 4.2: indeed, observe that for $(\boldsymbol{q}, \boldsymbol{q}^*, p) \in D^{\mathrm{c}}$, with

$$D^{\mathrm{c}} := \{(\boldsymbol{q}, \boldsymbol{q}^*, p) \in \mathbb{R}^5 : F_+(\boldsymbol{q}, \boldsymbol{q}^*, p) > 0\}$$

we have the limit result for the partition function

$$\lim_{n\to\infty} -\log Z_n(\boldsymbol{q},\boldsymbol{q}^*,p) = -\log\left(\lim_{s\to 0-}(1-b^s)Z(s,\boldsymbol{q},\boldsymbol{q}^*,p)\right) := F_\infty(\boldsymbol{q},\boldsymbol{q}^*,p).$$

Here,

$$F_\infty(\boldsymbol{q},\boldsymbol{q}^*,p) := -\log\frac{z_-(\boldsymbol{q},\boldsymbol{q}^*,p)}{1-z_+(\boldsymbol{q},\boldsymbol{q}^*,p)}.$$

Define $\tau_\infty(\boldsymbol{q},\boldsymbol{q}^*)$ by

$$F_\infty(\boldsymbol{q},\boldsymbol{q}^*,-\tau_\infty(\boldsymbol{q},\boldsymbol{q}^*))=0.$$

We have explicitly

$$\tau_\infty(\boldsymbol{q},\boldsymbol{q}^*) = -\log_m(z_-(\boldsymbol{q},\boldsymbol{q}^*)+z_+(\boldsymbol{q},\boldsymbol{q}^*)).$$

We shall then call $f_\infty(\alpha,\alpha^*)$, with

$$f_\infty(\alpha,\alpha^*) := \inf_{(\boldsymbol{q},\boldsymbol{q}^*)\in\mathbb{R}^4}(\langle\alpha,\boldsymbol{q}\rangle+\langle\alpha^*,\boldsymbol{q}^*\rangle-\tau_\infty(\boldsymbol{q},\boldsymbol{q}^*))$$

the Hölder spectrum of the regular part of the (A,B)-path graph.

5.3.5. Concentrating on the length of the (A,B)-path trail. This geometrical information is quite important: suppose one is, for example, interested in the *length* of the (A,B)-path trail, in the *confined* case. Then, one should focus only on the moduli of the available chunks, dropping the information on the signs.

Let then $f(\alpha_1,\alpha_1^*)$ be the Legendre transform of $\tau(q_1,q_1^*) = -\log_m z_+(q_1,q_1^*)$ with

$$z_+(q_1,q_1^*) := \sum_{l=d+1}^{m}|\pi_{l,+}|^{q_1}|r_{l,+}|^{q_1^*} \tag{5.19}$$

setting $q_2=q_2^*=0$ into $z_+(\boldsymbol{q},\boldsymbol{q}^*)$.

In the confined case, the Legendre variables (α_1,α_1^*) vary in the polygonal convex hull of the points $(-\log_m|\pi_{l,+}|,-\log_m|r_{l,+}|)_{l=d+1,\dots,m}$ and are thus positive.

Define the set

$$B(\alpha_1,\alpha_1^*) = \left\{i\in[N_n]:\frac{-\log|\pi_i(n)|}{\log N_n}\to\alpha_1;\frac{-\log|r_i(n)|}{\log N_n}\to\alpha_1^*\right\}$$

and let $N_n(\alpha_1,\alpha_1^*) := \#B(\alpha_1,\alpha_1^*)$ be its cardinal. For the atoms within $B(\alpha_1,\alpha_1^*)$: $|\pi_i(n)| \underset{n\to\infty}{\sim} N_n^{-\alpha_1}$ and $|r_i(n)| \underset{n\to\infty}{\sim} N_n^{-\alpha_1^*}$. Therefore, the modulus

$$\|X_i(n)\|_2 := (\pi_i(n)^2+r_i(n)^2)^{1/2}$$

of these atoms grows like $\|X_i(n)\|_2 \underset{n\to\infty}{\sim} N_n^{-\min(\alpha_1,\alpha_1^*)}$, and there are $N_n(\alpha_1,\alpha_1^*) \underset{n\to\infty}{\sim} N_n^{f(\alpha_1,\alpha_1^*)}$ such atoms.

Consider now the moduli partition function, for any real number δ:

$$M_n(\delta) := \sum_{i=1}^{N_n}(\pi_i(n)^2+r_i(n)^2)^{\delta/2}.$$

Observe that $l_n := M_n(1)$ is the length of the (A,B)-path trail, at resolution n.

The contribution to $M_n(\delta)$ of the atoms of $B(\alpha_1,\alpha_1^*)$ grows like

$$M_n(\delta,\alpha_1,\alpha_1^*) \underset{n\to\infty}{\sim} N_n^{f(\alpha_1,\alpha_1^*)}N_n^{-\delta\min(\alpha_1,\alpha_1^*)}.$$

It is the product of their numbers $N_n^{f(\alpha_1,\alpha_1^*)}$ by their '*modulus*$^\delta$' $N_n^{-\delta\min(\alpha_1,\alpha_1^*)}$.

Letting $\alpha_1 := \alpha\alpha_1^*$ and $f_\alpha(\alpha_1^*) := f(\alpha\alpha_1^*, \alpha_1^*)$, we also have

$$M_n(\delta, \alpha, \alpha_1^*) \underset{n\to\infty}{\sim} N_n^{f_\alpha(\alpha_1^*)-\delta\min(\alpha\alpha_1^*,\alpha_1^*)}. \tag{5.20}$$

If $\alpha < 1$, this reduces to $M_n(\delta, \alpha, \alpha_1^*) \underset{n\to\infty}{\sim} N_n^{f_\alpha(\alpha_1^*)-\delta\alpha\alpha_1^*}$. The maximal contribution to $M_n(\delta)$ is attained at $\alpha_{1,\mathrm{o}}^* = \alpha_{1,\mathrm{o}}^*(\delta, \alpha)$ defined by $f'_\alpha(\alpha_{1,\mathrm{o}}^*) = \delta\alpha$. At points $\alpha_1^* = \alpha_{1,\mathrm{o}}^*$, the quantity $f_\alpha(\alpha_1^*) - \delta\alpha\alpha_1^*$ is strictly positive or negative. This quantity vanishes at the unique point $\alpha_{1,\mathrm{o}}^* := \alpha_{1,\mathrm{o}}^*(\alpha)$ defined by $f'_\alpha(\alpha_{1,\mathrm{o}}^*) = l(\alpha)$, with $l(\alpha)$ the Legendre transform of the function $\tau_1^*(q_1)$ defined as usual by

$$\sum_{l=d+1}^{m} |\pi_{l,+}|^{q_1} |r_{l,+}|^{-\tau_1^*(q_1)} = 1.$$

This defines a unique $\delta := \delta(\alpha)$ by $\alpha\delta(\alpha) = l(\alpha)$.

If $\alpha \geqslant 1$, $M_n(\delta, \alpha, \alpha_1^*) \underset{n\to\infty}{\sim} N_n^{f_\alpha(\alpha_1^*)-\delta\alpha_1^*}$. The unique $\delta(\alpha)$ under interest is now defined by $\delta(\alpha) = l(\alpha)$.

Putting all this together, we get

$$\delta(\alpha) = l(\alpha)/\min(\alpha, 1) \tag{5.21}$$

and

$$M_n(\alpha) := M_n(\delta(\alpha), \alpha, \alpha_{1,\mathrm{o}}^*(\alpha)) \underset{n\to\infty}{\sim} 1.$$

We shall call $\delta(\alpha)$ the geometrical α-dimension of the (A, B)-path trail. It is the geometrical multifractal spectrum associated to the set of atoms

$$B(\alpha\alpha_{1,\mathrm{o}}^*(\alpha), \alpha_{1,\mathrm{o}}^*(\alpha)) = \left\{ i \in [N_n] : \frac{-\log|\pi_i(n)|}{-\log|r_i(n)|} \to \alpha, -\log_{N_n}|r_i(n)| \to \alpha_{1,\mathrm{o}}^*(\alpha) \right\}.$$

The contribution to $M_n(\delta)$ of $M_n(\alpha)$ is maximal and one. It is neither zero nor infinity and the set $B(\alpha\alpha_{1,\mathrm{o}}^*(\alpha), \alpha_{1,\mathrm{o}}^*(\alpha))$ carries all the information '*modulus*$^\delta$'.

The contribution to the total length $l_n := M_n(1)$ of the atoms within set $B(\alpha\alpha_{1,\mathrm{o}}^*(\alpha), \alpha_{1,\mathrm{o}}^*(\alpha))$ is

$$l_n(\alpha) := M_n(1, \alpha, \alpha_{1,\mathrm{o}}^*(\alpha)).$$

It follows from the above results that

$$l_n(\alpha) \underset{n\to\infty}{\sim} N_n^{\zeta(\alpha)} \tag{5.22}$$

with $\zeta(\alpha) = \min(\alpha, 1)\alpha_{1,\mathrm{o}}^*(\alpha)(\delta(\alpha) - 1)$. For those α for which the α-dimension $\delta(\alpha) > 1$, $l_n(\alpha)$ diverges, as conventional wisdom suggests.

Remark 10. A similar study is possible on the θ-partition function defined by

$$S_n(\theta) := \sum_{i=1}^{N_n} (\mathrm{e}^{2b(\pi_i(n))} + \mathrm{e}^{2b(r_i(n))})^{\theta/2}$$

which focuses on the number of occurrences of negative elementary reduction factors

$$\left(b(\pi_i(n)) = \sum_{j=1}^{n} b(\pi_{l_j(i)}), b(r_i(n)) = \sum_{j=1}^{n} b(r_{l_j(i)}) \right)$$

in the unique multiplicative decomposition of each of the increments

$$\left(\pi_i(n) := \prod_{j=1}^{n} \pi_{l_j(i)}, r_i(n) := \prod_{j=1}^{n} r_{l_j(i)} \right).$$

6. Concluding remarks

This work is an attempt towards generalizing the usual construction and analysis of multifractal measures in two directions. The first consists in considering mixtures of absolutely continuous and purely singular measures through an iterating function system (IFS) with 'sterile' parts. Measures with atoms or gaps are easily introduced as limiting situations in this framework. We have proceeded with the thermodynamical study of these sterile parts in the IFS, using a Gibbs formalism.

Then, we have bridged the gap between measures towards functions and more generally towards 'bridges', while allowing the 'reduction factors' attributed to the maps of the IFS to be negative. It should be noted that the relation to [29, 30] is through the special case where the reduction factors in one dimension of space are positive numbers, whereas those in other dimensions of space are allowed to be real numbers, but summing up to one. Then, a bridge is the graph of a self-similar function or, in other words, the graph of the distribution function of a one-dimensional vector-valued self-similar measure. As in [30], this work generalizes thus self-affine functions [29] but in a different direction than [30]: not to measures in higher-dimensional spaces but to more general (A, B)-bridges or paths. Consequently, the approach developed in this work is different in the sense that it focuses on the properties of the object as a subset of the plane, and not as a real-valued measure on the line.

For such paths, the main lines of the multifractal formalism have been derived. These paths are allowed to be constituted with both macroscopic and microscopic fragments, each deserving its specific statistics, while remaining continuous, exploiting the ideas developed in the first part of the work on measures. Discontinuous paths may also be considered with some information related to the statistics of their jumps.

References

[1] Halsey T, Jensen M, Kadanoff L, Procaccia I and Shraiman B 1986 Fractal measures and their singularities. The characterization of strange sets *Phys. Rev.* A **33** 1141–51

[2] Mandelbrot B 1974 Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier *J. Fluid Mech.* **62** 331–58

[3] Frisch U and Parisi G 1985 On the singularity structure of fully developed turbulence *Proc. Int. School Phys., 'Enrico Fermi' Course LXXXVIII, Varenna, 1983* (Amsterdam: North-Holland)

[4] Mandelbrot B 1990 New 'anomalous' multiplicative multifractals: left-sided $f(\alpha)$ and the modelling of DLA *Physica* **168A** 95–111

[5] Mandelbrot B and Evertsz C 1991 Multifractality of the harmonic measure on fractal aggregates, and extended self-similarity *Physica* **177A** 386–93

[6] Mandelbrot B 1992 Multifractal measures *Chaos and Fractals, New frontiers of Science* ed H O Peitgen, H Jürgens and D Saupe (Berlin: Springer)

[7] Ellis R S 1985 *Entropy, Large Deviations and Statistical Mechanics* (New York: Springer)

[8] Huillet T and Jeannet B 1994 Random walk models for multifractals *J. Phys. A: Math. Gen.* **27** 6315–34

[9] Mandelbrot B and Riedi R 1997 Inverse measures, the inversion formula, and discontinuous multifractals *Adv. Appl. Math.* **18** 50–8

[10] Riedi R and Mandelbrot B 1997 Exceptions to the multifractal formalism for discontinuous measures *Math. Proc. Cambr. Phil. Soc.* to appear

[11] Riedi R and Mandelbrot B 1997 The inversion formula for continuous multifractals *Adv. Appl. Math.* **19** 332–54

[12] Mandelbrot B, Evertsz C and Hayakawa Y 1990 Exactly self-similar left-sided multifractal measures *Phys. Rev.* A **42** 4528–36

[13] Riedi R and Mandelbrot B 1995 Multifractal formalism for infinite multinomial measures *Adv. Appl. Math.* **16** 132–50

[14] Mandelbrot B 1995 Negative dimensions and Hölders, multifractals and their Hölder spectra, and the role of lateral preasymptotics in science *J. Fourier Anal. Applic. (Kahane Special Issue)* 410–32
[15] Mandelbrot B 1990 Negative fractal dimensions and multifractals *Physica* **163A** 306–15
[16] Mandelbrot B 1989 A class of multinomial multifractal measures with negative (latent) values for the 'dimension' $f(\alpha)$ *Fractals' Physical Origin and Properties* ed L Pietronero (New York: Plenum) pp 3–29
[17] Mandelbrot B 1989 Limit lognormal multifractal measures *Frontiers of Physics: Landau Memorial Conf. (Tel Aviv, 1988)* ed E Gotsman (New York: Pergamon)
[18] Huillet T and Jeannet B 1996 Finitely generated multifractals can display phase transitions *J. Physique* I **6** 245–56
[19] Huillet T and Jeannet B 1994 On a class of "skewed" self-similar and hyperbolic fractals *J. Math. Phys.* A **35** 6511–24
[20] Brown G, Michon G and Peyrière J 1992 On the multifractal analysis of measures *J. Stat. Phys.* **66** 775–90
[21] Mendès France M and Tenenbaum G 1993 A one dimensional model with phase transition *Commun. Math. Phys.* **154** 603–11
[22] Arbeiter M and Patzschke N 1996 Random self-similar multifractals *Math. Nachr.* **181** 5–42
[23] Rosenkrantz R D 1983 *Papers on Probability, Statistics and Statistical Physics* vol 158, ed E T Jayne (Dordrecht: Reidel)
[24] Levine R D and Tribus M (eds) 1979 *The Maximum Entropy Formalism* (Cambridge, MA: MIT Press)
[25] Hammad P 1987 *Information, systèmes et distributions* (Paris: Editions Cujas)
[26] Azencott R 1978 Grandes déviations et applications *Cours de Saint-Flour (Lecture Notes in Math. 774)* (Heidelberg: Springer)
[27] Jeannet B, Huillet T and Choukri K 1997 Marches pseudo-aléatoires issues de substitutions: aspects statistiques *Annal. Institut H Poincaré, Phys. Théor.* **67.1** 77–89
[28] Dekking F M 1991 Substitutions, branching processes and fractal sets *Fractal Geometry and Analysis (NATO ASI Series 346)* ed J Bélair and S Dubuc (Dordrecht: Kluwer)
[29] Jaffard S 1993 Multifractal formalism for functions *CRAS* **317** 745–50
[30] Falconer K J and O'Neil W 1996 Vector-valued multifractal measures *Proc. R. Soc.* A **452** 1433–57